MATH-4630-01 **Homework Assignment #1**

DUE: *Friday, Jan 30, 11:59p* Sections 1.1–1.5

Instructions: Show all work clearly and justify each conclusion. Collaboration is encouraged, but *write up your solutions individually in your own words.* For any **prove/disprove** problem: either give a proof, or give a specific counterexample (with a brief explanation of why it works).

Problem 1. Let $A, B \subseteq \mathbb{R}$ be nonempty and bounded.*

(a) Prove that if $A \subseteq B$, then $\sup A \leq \sup B$.

(b) **Prove or disprove:** If $\sup A < \inf B$, then there exists a real number c such that

$$a < c < b \quad \text{for all } a \in A \text{ and } b \in B.$$

Problem 2. Let $a, b \in \mathbb{R}$. Recall the triangle inequality:

$$|x + y| \leq |x| + |y| \qquad \text{for all } x, y \in \mathbb{R}.$$

(a) Use the identity

$$a = (a - b) + b$$

and apply the triangle inequality to obtain an upper bound for $|a|$ in terms of $|a - b|$ and $|b|$.

(b) Interchange the roles of a and b to obtain a similar inequality for $|b|$.

(c) Deduce that

$$\big||a| - |b|\big| \leq |a - b|.$$

Problem 3. Let E and F be nonempty subsets of $\mathbb{R}$ that are bounded above, and define

$$E + F = \{\, x + y : x \in E \text{ and } y \in F \,\}.$$

(a) Let $\alpha = \sup E$ and $\beta = \sup F$. Explain why α and β exist, and show that $\alpha + \beta$ is an upper bound for $E + F$.

(b) Let M be an arbitrary upper bound for $E + F$, and fix $x \in E$. Show that

$$\beta \leq M - x.$$

(c) Conclude that

$$\sup(E + F) = \alpha + \beta.$$

Problem 4. For each set below, determine and **justify** $\sup A$ and $\inf A$, and state whether they belong to the set.

(a) $A = \{x \in \mathbb{R} : x^2 < 2\}$ (b) $A = \{1 - 1/n : n \in \mathbb{N}\}$

Problem 5. Recall that a set is called *countable* if it is finite or can be put into a one-to-one correspondence with $\mathbb{N}$.

(a) Prove that any finite subset of $\mathbb{R}$ is countable.

(b) Show that the set

$$A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$$

is countable by explicitly describing a bijection with $\mathbb{N}$.

(c) Let $B = \mathbb{Z} \cap [0, \infty)$. Prove that B is countable.

(d) Prove that the union of a finite set and a countable set is countable.

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*A set is bounded if it is both bounded above and bounded below.