MATH-4630-01 **Homework Assignment #2**

DUE: *Tuesday, Feb 17, 11:59p* Sections 2.1–2.5

Instructions: Show all work clearly and justify each conclusion. Collaboration is encouraged, but *write up your solutions individually in your own words.* For any **prove/disprove** problem: either give a proof, or give a specific counterexample (with a brief explanation of why it works).

Problem 1. Using the definition of convergence, prove the following limits.

(a) $\lim_{n\to\infty}\left(\frac{3n+1}{2n+5}\right)=\frac{3}{2}$. (b) $\lim_{n\to\infty}\left(\frac{n}{n^2+1}\right)=0$.

Problem 2. Give an example of each or state that it is impossible. For any that are impossible, give a compelling argument for why that is the case.

(a) A sequence with an infinite number of ones that does not converge to one.
(b) A sequence with an infinite number of ones that converges to a limit not equal to one.
(c) A divergent sequence such that for every $n \in \mathbb{N}$ it is possible to find n consecutive ones somewhere in the sequence.

Problem 3. Let $(a_n) \to 0$, and use the Algebraic Limit Theorem to compute each of the following limits (assuming the fractions are always defined):

(a) $\lim_{n\to\infty}\left(\frac{1+2a_n}{1+3a_n-4a_n^2}\right)$ (b) $\lim_{n\to\infty}\left(\frac{(a_n+2)^2-4}{a_n}\right)$

Problem 4. Give an example of each of the following, or state that such a request is impossible by referencing the proper theorem(s):

(a) sequences (x_n) and (y_n), which both diverge, but whose sum (x_n+y_n) converges;
(b) sequences (x_n) and (y_n), where (x_n) converges, (y_n) diverges, and (x_n+y_n) converges;
(c) a convergent sequence (b_n) with $b_n \neq 0$ for all n such that $(1/b_n)$ diverges;
(d) an unbounded sequence (a_n) and a convergent sequence (b_n) with $(a_n - b_n)$ bounded;
(e) two sequences (a_n) and (b_n), where (a_nb_n) and (a_n) converge but (b_n) does not.

Problem 5. Prove that the sequence defined by $x_1 := 1$ and

$$x_{n+1} := \sqrt{2+x_n}, \quad \text{for all } n \geq 1$$

is convergent. Find the limit.

Problem 6. Let (a_n) be a bounded sequence.

(a) Prove that the sequence defined by $y_n = \sup\{a_k : k \geq n\}$ converges.
(b) The *limit superior* of (a_n), or $\limsup_{n\to\infty} a_n$, is defined by

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} y_n,$$

where (y_n) is the sequence from part (a) of this exercise. Similarly, define $z_n = \inf\{a_k : k \geq n\}$. The *limit inferior* of (a_n), or $\liminf_{n\to\infty} a_n$, is defined by

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} z_n.$$

Prove that the sequence (z_n) converges, and briefly explain why $\liminf_{n\to\infty} a_n$ always exists for any bounded sequence.

(c) Prove that $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$ for every bounded sequence, and give an example of a sequence for which the inequality is strict.

(d) Show that $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$ if and only if $\lim_{n\to\infty} a_n$ exists. In this case, all three share the same value.

Problem 7. Give an example of each of the following, or argue that such a request is impossible.

(a) A sequence that has a subsequence that is bounded but contains no subsequence that converges.

(b) A sequence that does not contain 0 or 1 as a term but contains subsequences converging to each of these values.

(c) A sequence that contains subsequences converging to every point in the infinite set $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\}$.

(d) A sequence that contains subsequences converging to every point in the infinite set $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\}$, and no subsequences converging to points outside of this set.

Problem 8. Decide whether the following propositions are true or false, providing a short justification for each conclusion.

(a) If every proper subsequence of (x_n) converges, then (x_n) converges as well.

(b) If (x_n) contains a divergent subsequence, then (x_n) diverges.

(c) If (x_n) is bounded and diverges, then there exist two subsequences of (x_n) that converge to different limits.

(d) If (x_n) is monotone and contains a convergent subsequence, then (x_n) converges.

Problem 9 (Bonus problem:). Suppose that every subsequence of a sequence (x_n) has a further subsequence that converges to 0. Prove that

$$\lim_{n\to\infty} x_n = 0.$$

(**Hint:** Prove using contradiction!)
