MATH-4630-01 **Homework Assignment #3**

DUE: *Friday, Mar 6, 11:59p* Sections 2.7, 3.2

Instructions: Show all work clearly and justify each conclusion. Collaboration is encouraged, but *write up your solutions individually in your own words.*
For any **prove/disprove** problem: either give a proof, or give a specific counterexample (with a brief explanation of why it works).

Problem 1. Decide whether each of the following series converges or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n + n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$

(c) $1 - \frac{3}{4} + \frac{4}{6} - \frac{5}{8} + \frac{6}{10} - \frac{7}{12} + \cdots$

(d) $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \cdots$

(e) $1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \frac{1}{5} - \frac{1}{6^2} + \frac{1}{7} - \frac{1}{8^2} + \cdots$

Problem 2. Give an example of each or explain why the request is impossible referencing the proper theorem(s).

(a) Two series $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ that both diverge but where $\sum_{n=1}^{\infty} x_n y_n$ converges.

(b) A convergent series $\sum_{n=1}^{\infty} x_n$ and a bounded sequence (y_n) such that $\sum_{n=1}^{\infty} x_n y_n$ diverges.

(c) Two sequences (x_n) and (y_n) where $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty}(x_n + y_n)$ both converge but $\sum_{n=1}^{\infty} y_n$ diverges.

(d) A sequence (x_n) satisfying $0 \le x_n \le 1/n$ where $\sum_{n=1}^{\infty}(-1)^n x_n$ diverges.

Definition 1. A series $\sum_{n=1}^{\infty} a_n$ is said to **converge conditionally** if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

Problem 3. Consider each of the following propositions. Provide short proofs for those that are true and counterexamples for any that are not.

(a) If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n^2$ also converges absolutely.

(b) If $\sum_{n=1}^{\infty} a_n$ converges and (b_n) converges, then $\sum_{n=1}^{\infty} a_n b_n$ converges.

(c) If $\sum_{n=1}^{\infty} a_n$ converges conditionally, then $\sum_{n=1}^{\infty} n^2 a_n$ diverges.

Problem 4. Decide whether the following statements are true or false. Provide counterexamples for those that are false, and supply proofs for those that are true.

(a) An open set that contains every rational number must necessarily be all of $\mathbb{R}$.

(b) Every nonempty open set contains a rational number.

(c) Every bounded infinite closed set contains a rational number.

Problem 5. Assume A is an open set and B is a closed set. Determine if the following sets are definitely open, definitely closed, both, or neither.

(a) $\overline{A \cup B}$

(b) $A \setminus B = \{x \in A : x \notin B\}$

(c) $(A^c \cup B)^c$

(d) $(A \cap B) \cup (A^c \cap B)$

(e) $\overline{A}^c \cap \overline{A^c}$

Problem 6.

(a) Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(b) Does this result about closures extend to infinite unions of sets?
