**MATH-4630-01** **Homework Assignment #4**

DUE: *Friday, Mar 27, 11:59p* Sections 3.3, 3.4

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**Instructions:** Show all work clearly and justify each conclusion. Collaboration is encouraged, but *write up your solutions individually in your own words.*
For any **prove/disprove** problem: either give a proof, or give a specific counterexample (with a brief explanation of why it works).

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**Problem 1.** For each set $K$, determine whether it is compact. If $K$ is not compact, produce a sequence $(x_n) \subseteq K$ such that *every* convergent subsequence of $(x_n)$ converges to a limit not contained in $K$. If $K$ is compact, explain why every sequence in $K$ has a subsequence converging to a limit in $K$. **Do not use Heine-Borel theorem for this problem!**

**(a)** $\mathbb{N}$.

**(b)** $\left\{1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots\right\}$.

**(c)** $\left\{1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} : n \in \mathbf{N}\right\}$.

$\left(\textbf{Hint:} \text{ You may have to use the fact that } \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}\right)$.

**Problem 2.** Assume $K$ is compact and $F$ is closed. For each of the following sets, decide whether it is definitely compact, definitely closed, both, or neither. In each case, *explicitly argue* your conclusion using any method (e.g., sequences, open covers, or the Heine–Borel Theorem).

**(a)** $K \cap F$.

**(b)** $\overline{F^c \cup K^c}$.

**(c)** $K \setminus F = \{x \in K : x \notin F\}$.

**(d)** $\overline{K \cap F^c}$.

**Problem 3.** Let $A$ and $B$ be nonempty subsets of $\mathbb{R}$. Show that if there exist disjoint open sets $U$ and $V$ with $A \subseteq U$ and $B \subseteq V$, then $A$ and $B$ are separated.

(**Hint:** As an intermediate step, first show that $\overline{A} \subseteq U$ and $\overline{B} \subseteq V$. Once you have this, the conclusion follows quickly from $U \cap V = \emptyset$.)

**Problem 4.** Let $K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$ be a nested sequence of nonempty compact sets. Show that the intersection

$$\bigcap_{n=1}^{\infty} K_n$$

is nonempty.

(**Hint:** For each $n \in \mathbf{N}$, choose a point $x_n \in K_n$ and consider the sequence $(x_n)$.)

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