Construction of $\mathbb{R}$ — The Axiom of Completeness (Part 1)

The real numbers $\mathbb{R}$ are fundamentally one of the important objects for real analysis. Although we use real numbers constantly, it is not immediately clear which of their properties should be taken as assumptions and which should be proved. In this handout, we will describe the real number system *axiomatically.* Rather than constructing $\mathbb{R}$ from scratch, we will list a small number of fundamental properties, called *axioms*—that characterize the real numbers and distinguish them from other familiar number systems. This approach, developed largely in the nineteenth century, allows us to understand precisely what makes the real numbers special and why certain results of calculus and analysis are possible.

Axiom 1: [Field Axioms]

Let $\mathbb{R}$ be a set. We assume there exist two binary operations on $\mathbb{R}$, called $+$ (addition) and $\cdot$ (multiplication), such that $(\mathbb{R}, +, \cdot)$ is a ______________. This means that for all $x, y, z \in \mathbb{R}$:

- **Closure:** $x + y \in \mathbb{R}$ and $x \cdot y \in \mathbb{R}$.
- **Associativity:** $(x + y) + z = x + (y + z)$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.
- **Commutativity:** $x + y = y + x$, $x \cdot y = y \cdot x$.
- **Identities:** There exist $0, 1 \in \mathbb{R}$ such that $x + 0 = x$ and $x \cdot 1 = x$.
- **Inverses:**
  - (Additive inverse) For every x, there exists $-x$ such that $x + (-x) = 0$.
  - (Multiplicative inverse) For every $x \neq 0$, there exists x^{-1} such that $x \cdot x^{-1} = 1$.
- **Distributive Law:** $x(y + z) = xy + xz$.

We *define* subtraction and division by

$x - y =$ __________________________ and $x \div y =$ ______________ $(y \neq 0)$.

Fact: For each $x \in \mathbb{R}$, the additive inverse is *unique*, and if $x \neq 0$, the multiplicative inverse is *unique*.

The "Wish List"

The following statements about real numbers are all true, but they are not explicitly listed in Axiom 1. Some will follow from Axiom 1, while others require additional axioms.

1. For all $x \in \mathbb{R}$, $x \cdot 0 = 0$.
2. For all $x \in \mathbb{R}$, $-x = (-1) \cdot x$.
3. For all nonzero real numbers a, $a \neq -a$.
4. $\mathbb{R}$ is an infinite set.
5. There exists a real number x such that $x^2 = 2$.

Exercise. Prove that for all $x \in \mathbb{R}$, $x \cdot 0 = 0$ using only the field axioms.

Axiom 2: [Order Axioms]

There exists a subset $\mathbb{R}^+ \subseteq \mathbb{R}$, called the set of ______________________________.
This set satisfies:

1. If $x, y \in \mathbb{R}^+$, then $x + y \in$ __________ and $x \cdot y \in$ __________.
2. For every real number a, *exactly one* of the following is true:
   - a is ____________________.
   - a is ____________________.
   - $-a$ is ____________________.

We call elements of $\mathbb{R}^+$ ______________________________. The set of _______________ real numbers is the complement of $\mathbb{R}^+ \cup \{0\}$ in $\mathbb{R}$, and we denote it by __________.

Exercise. Verify the following facts using Axiom 2.

1. The real number 1 is positive.
2. There exists a negative real number.
3. For all nonzero real numbers a, $a \neq -a$.

Note. The rational numbers $\mathbb{Q}$ satisfy both Axiom 1 and Axiom 2.

So what is missing? If $\mathbb{Q}$ already satisfies Axioms 1 and 2, why do we need $\mathbb{R}$?

Thought Experiment. There is *no* rational number whose square is 2.

So we can't hope that every positive real number has a square root using only *Axiom 1* and *Axiom 2*. We need one more axiom – an axiom that will "fill the holes" left by rational numbers:

Axiom 3: [The Axiom of Completeness]

Every ____________________ subset of $\mathbb{R}$ that is bounded above has a ____________________.

We will treat this as an ____________________: something we may use without proof.

In the next handout, we will unpack what "bounded above" and "least upper bound" mean, and we will learn how to compute $\sup A$ and $\inf A$ in examples.