The Axiom of Completeness – (Part 2)

In the previous handout, we stated the following axiom:

Axiom 3: Every *nonempty* subset of $\mathbb{R}$ that is *bounded above* has a *least upper bound.*

In this handout, we will explore Axiom 3 in details. Let us begin with some important definitions.

Definition:

A set $A \subseteq \mathbb{R}$ is **bounded above** if there exists $b \in \mathbb{R}$ such that

____

Such a number b is called an ______________________ for A.

A set $A \subseteq \mathbb{R}$ is **bounded below** if there exists $\ell \in \mathbb{R}$ such that

____

Such a number ℓ is called a ______________________ for A.

Exercise:

For each set below, complete the table.

Set	Bounded Above?	Bounded Below?	One Upper Bound	One Lower Bound
(i) $A = (0, 1)$				
(ii) $B = \{x \in \mathbb{R} : x^2 < 4\}$				
(iii) $C = \{x \in \mathbb{R} : x \geq 3\}$				
(iv) $D = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$				
(v) $E = \{x \in \mathbb{R} : x < 5\}$				

Definition:

A real number s is the **least upper bound** or **supremum** of A (written $s = \sup A$) if:

(i) s is an upper bound for A: ____

(ii) s is the *least* such bound: if b is any upper bound for A, then

____.

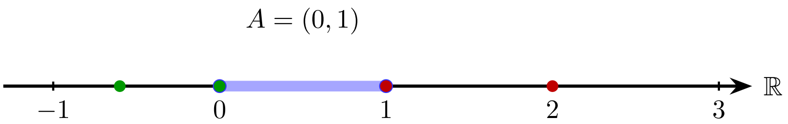
A real number ℓ is the **greatest lower bound** or **infimum** of A (written $\ell = \inf A$) if:

(i) ℓ is a lower bound for A: ____

(ii) ℓ is the *greatest* such bound: if m is any lower bound for A, then

____.

An illustration:



Exercise:

Let $A \subseteq \mathbb{R}$ be a nonempty set that is bounded above. Suppose s_1 and s_2 are both supremum of A. Prove that $s_1 = s_2$.

Note: The supremum of a set ________________ always exist.

- If a set is ____, then it has no supremum (e.g. $\mathbb{N}$).
- The empty set has ________________ supremum. By convention, one sometimes writes $\sup \varnothing = -\infty$, but this is not a real number and is outside our definition.

Exercise:

For each set below, determine $\sup A$ and $\inf A$ (if they exist) and whether they belong to the set.

(i) $A = (0,1)$

(ii) $C = \{x \in \mathbb{R} : x \geq 3\}$

(iii) $D = \{x \in \mathbb{R} : x < 5\}$

Reflection. Give an example from the list above where:

1. the supremum exists but *does not belong* to the set: ____________________________
2. the infimum exists but *does not belong* to the set: ____________________________
3. *neither* the supremum nor the infimum belongs to the set: ____________________________

Note:

In general, $\sup A$ and $\inf A$ *need not belong* to the set A.

- If $\sup A \in A$, then $\sup A$ is called the _________________ of A, and we write $\max A = \sup A$.
- If $\inf A \in A$, then $\inf A$ is called the _________________ of A, and we write $\min A = \inf A$.

We conclude this handout by proving an *equivalent* and useful characterization of the least upper bound (supremum). An analogous characterization holds for the infimum and will be assigned as homework.

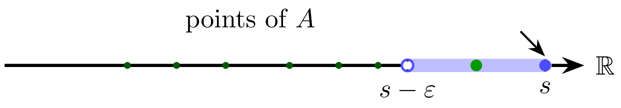
Proposition 1. *Let $A \subseteq \mathbb{R}$ be nonempty and bounded above, and assume s is an upper bound for A. Then $s = \sup A$ if and only if:*

For every $\varepsilon > 0$, there exists $a \in A$ such that

____.

Visually:

For any __________, the interval $(s - \varepsilon,\, s]$ must contain ____________________.



Takeaway: No matter how small ε is, A must come within ε of s from below.

Proof. Let $A \subseteq \mathbb{R}$ be nonempty and bounded above, and assume s is an upper bound for A. We prove both directions.

($\Rightarrow$) Assume $s = \sup A$. Let $\varepsilon > 0$. Since s is the _____________ upper bound of A, the number $s - \varepsilon$ cannot be an _____________ bound for A. Therefore, there exists $a \in A$ such that

____.

This proves the forward direction.

($\Leftarrow$) Assume: for every $\varepsilon > 0$, there exists $a \in A$ such that $s - \varepsilon < a \leq s$. We already know that s is an _____________ bound for A. To show $s = \sup A$, we must show that s is the ____________________ such bound.

Suppose for contradiction that there is an upper bound b of A with

__________________.

Define $\varepsilon :=$ __________________. Then there exists $a \in A$ such that

____.

Explain why this contradicts that b is an upper bound:

____

Hence no upper bound can be smaller than s, and so

$$s = \underline{\hspace{4cm}}.$$

□

Note:

1. We have not proved in this handout that there exists a real number x such that $x^2 = 2$ ("Wish List"–5). This fact will be proved in the next handout as a consequence of the completeness axiom (Axiom 3).
2. Through Axiom 2, everything would also work in the rational numbers $\mathbb{Q}$. The completeness axiom is the first assumption that genuinely distinguishes $\mathbb{R}$ from $\mathbb{Q}$.