Consequences of Completeness

In the previous handout, we established the **Axiom of Completeness** (Axiom 3).

Axiom 3: Every *nonempty* subset of $\mathbb{R}$ that is *bounded above* has a *least upper bound* (supremum) in $\mathbb{R}$.

In this section, we will explore four fundamental consequences of this Axiom. This handout is based on the first two consequences of Axiom 3. We will discuss consequences 3 and 4 in the next handout.

1. **Archimedean Property:** Natural numbers grow without bound
2. **Nested Intervals:** Shrinking intervals converge to a point
3. **Density of $\mathbb{Q}$:** Rationals are everywhere in $\mathbb{R}$
4. **Existence of $\sqrt{2}$:** We can finally prove it exists! ("Wish List" item V)

Note: All four properties *fail* in $\mathbb{Q}$! Completeness is what makes them work.

1. The Archimedean Property

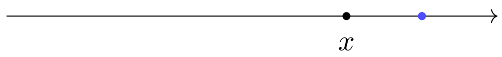
Theorem 1 (Archimedean Property)**.** ***(i)*** *Given any real number $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that*

________________________________.

(ii) *Given any real number $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that*

________________________________.

Intuition:
Part (i): No matter how large x is, we can find a natural number bigger than x.
The natural numbers are ______________________ — they have no upper bound.



Part (ii): No matter how small $\varepsilon > 0$ is, we can find $\frac{1}{n}$ smaller than ε.
We can make $\frac{1}{n}$ ____________________________.

Proof of part (i). Suppose for contradiction that $\mathbb{N}$ is bounded above. Then by Axiom 3, $\mathbb{N}$ has a ____; call it $s = \sup \mathbb{N}$.

Since s is the least upper bound, $s - 1$ is ________________ an upper bound for $\mathbb{N}$.
Therefore, there exists $n \in \mathbb{N}$ such that: _________________________________

But then $n + 1 \in \mathbb{N}$ and: _________________________________

This contradicts that s is an ___________________________ for $\mathbb{N}$.

Therefore, $\mathbb{N}$ is ___________________________. Hence for any $x \in \mathbb{R}$, $\exists\, n \in \mathbb{N}$ with $n > x$. $\square$

Exercise:

1. Use part (i) to prove part (ii).

   Hint: Apply part (i) to $x = \frac{1}{\varepsilon}$.

2. **True or False:** For any $x > 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < x$.

3. Find $n \in \mathbb{N}$ such that $\frac{1}{n} < 0.001$.

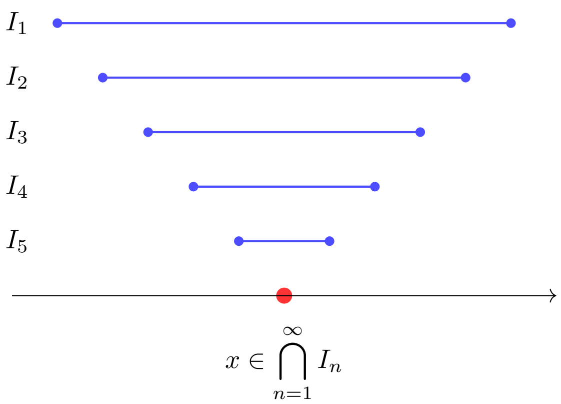
2. The Nested Interval Property

Theorem 2 (Nested Interval Property). *Assume $I_n = [a_n, b_n]$ is a sequence of closed, bounded intervals with*

____

Then the intersection $\bigcap_{n=1}^{\infty} I_n$ *is* ____________________.

Visualally:



Proof sketch. Define the set

$$A = \{a_n : n \in \mathbb{N}\} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Step 1: Show that A is *non-empty* and *bounded above.*

Step 2: By Axiom 3, let $x =$ ________________________.

Step 3: Prove that $x \in I_n$ for all $n \in \mathbb{N}$:

- Show $a_n \leq x$ for all $n \in \mathbb{N}$:

- Show $x \leq b_n$ for all $n \in \mathbb{N}$:

Altogether then, we have ________________________ and, therefore $x \in \bigcap_{n=1}^{\infty} I_n$, so the intersection is nonempty. □

Exercise:

Consider the sequence of intervals

$$I_n = [0, \tfrac{1}{n}] \quad \text{for } n = 1, 2, 3, \ldots$$

1. Verify that:

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$$

2. What is

$$\bigcap_{n=1}^{\infty} I_n \ = ?$$

3. Now consider the nested *open* intervals

$$J_n = (0, \tfrac{1}{n}) \quad \text{for } n = 1, 2, 3, \ldots$$

   (a) Show that $J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots$.

   (b) Prove that

$$\bigcap_{n=1}^{\infty} J_n = \emptyset.$$

4. Explain why the Nested Interval Property holds for (I_n) but fails for (J_n).