Consequences of Completeness (Part 2)

In the previous handouts, we established the Archimedean Property and the Nested Interval Property as consequences of the Axiom of Completeness. In this handout, we focus on two further consequences that highlight *approximation* and *existence* in $\mathbb{R}$. We will start with a Proposition (wihtout proof) that will be useful to prove the density of $\mathbb{Q}$.

Proposition 1. *Every nonempty subset of $\mathbb{R}$ that is bounded below has a greatest lower bound (infimum) in $\mathbb{R}$.*

Proof. The proof is left as an Exercise! I will provide some hints.

Hint. Let $A \subset \mathbb{R}$ be a nonempty set that is bounded below.

Consider the set

$$-A = \{ -x : x \in A \}.$$

(i) Show that $-A$ is *nonempty* and *bounded above*. (Hint: If m is a lower bound for A, what is an upper bound for $-A$?)

(ii) By the Axiom 3, the set $-A$ has a supremum. Let

$$s = \sup(-A).$$

(iii) Show that $-s$ is a *lower bound* for A.

(iv) Show that $-s$ is the *greatest lower bound* for A.

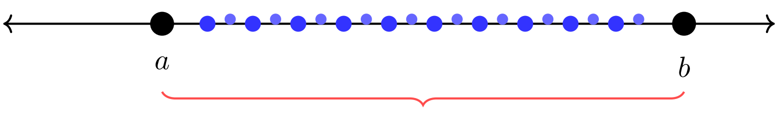
Conclude that $-s = \inf A$. □

3. The Density of $\mathbb{Q}$ in $\mathbb{R}$

Theorem 1 (Density of $\mathbb{Q}$)**.** *For any two real numbers $a < b$, there exists a rational number* __________ *such that*

____

No matter how close two real numbers are, we can always find a rational number strictly between them. This says that $\mathbb{Q}$ has *no gaps* inside $\mathbb{R}$.



Proof sketch: **Step 1:**
Because $a < b$, the difference $b - a$ is __________.
By the Archimedean Property, there exists $n \in \mathbb{N}$ such that

$$\frac{1}{n} < \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_}.$$

Step 2:
Multiply the inequality $\frac{1}{n} < b - a$ by n to obtain

$$\_\_\_\_\_\_\_\_\_\_ < n(b-a).$$

Explain *why* the inequality $na < na + 1 < nb$ is true:

____

Define the set:

$$S = \{\, k \in \mathbb{Z} : k > na \,\}.$$

1. Explain why S is *nonempty.*

2. Is S bounded below? ☐ Yes ☐ No. Justify your choice.

From Proposition 1, the set S has a ________________ in $\mathbb{R}$. Let m be the ________________ of S.

Step 3:
Define

$$r = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_.$$

Explain why r is a rational number:

Step 4:
From the definition of m, we have

$$na < m.$$

Dividing by n, conclude that

$$\_\_\_\_\_\_\_\_\_\_ < \frac{m}{n} = r.$$

Use the minimality of m to show that $m \leq na + 1$. (**Hint:** What can you say about $m - 1$? Is it in S?)

Divide by n and use Step 1 to conclude

$$r \; < \; \underline{\hspace{3cm}}.$$

Therefore,

$$a \; < \; r \; < \; b.$$

□

4. Existence of n^{th} Roots

This is the moment we've been waiting for! We can now prove that $\sqrt{2}$ exists– "Wish List–(v)".

Theorem 2. *There exists a real number $\alpha \in \mathbb{R}$ such that $\alpha^2 = 2$.*
We denote this number by $\alpha = \sqrt{2}$ (or $2^{1/2}$).

Proof sketch: Define the set

$$S = \{s \in \mathbb{R} : s^2 < 2\}.$$

Explain why the set S is nonempty and bounded above:

Therefore by Axiom _____, the set S has a _______________________________________ in $\mathbb{R}$, call it $x \in \mathbb{R}$.

Explain why $x > 1$:

Since the real numbers satisfy the *Trichotomy Law*, so exactly one of the following must hold:

$$x^2 < 2, \quad x^2 = 2, \quad x^2 > 2.$$

The strategy is to show that $x^2 = 2$ by ruling out the other following possibilities.

Case 1: Assume $x^2 < 2$.

Our goal is to find a number *slightly larger than* x that still belongs to S.

Explain why this would contradict the fact that $x = \sup S$:

Consider: $\left(x+\frac{1}{n}\right)^2 = x^2 + \_\_\_\_\_\_\_\_ + \_\_\_\_\_\_\_\_.$

Show that: $\left(x+\frac{1}{n}\right)^2 \leq x^2 + \frac{1}{n}(2x+1)$.

We want: $\left(x+\frac{1}{n}\right)^2 < 2$. This will happen provided

$$\frac{1}{n}(2x+1) < \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_.$$

Explain why the expression $\frac{2-x^2}{2x+1}$ is positive: ____.

By the Archimedean Property, we can choose $n \in \mathbb{N}$ such that

$$\frac{1}{n} < \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_.$$

Conclude that $x + \frac{1}{n} \in S$, and explain why this is a contradiction:

Therefore, we cannot have $x^2 < 2$.

Case 2: Assume $x^2 > 2$.

This time, we try to find a number *slightly smaller than* x that is still an upper bound for S.

Explain why this would contradict the definition of supremum:

Compute: $\left(x-\frac{1}{m}\right)^2 = x^2 - \_\_\_\_\_\_\_\_ + \_\_\_\_\_\_\_\_.$

Show that: $\left(x-\frac{1}{m}\right)^2 > x^2 - \frac{2x}{m}$.

We want $\left(x - \frac{1}{m}\right)^2 > 2$. This will occur if

$$\frac{2x}{m} < \underline{\hspace{3cm}}.$$

Explain why the expression $\frac{x^2 - 2}{2x}$ is positive: ____.

Use the Archimedean Property to choose $m \in \mathbb{N}$ such that

$$\frac{1}{m} < \underline{\hspace{6cm}}.$$

Explain why $\left(x - \frac{1}{m}\right)$ is then an upper bound for S, and why this is a contradiction:

Therefore, we cannot have $x^2 > 2$.

Conclusion. Since both $x^2 < 2$ and $x^2 > 2$ lead to contradictions, we conclude

$$x^2 = \underline{\hspace{1.5cm}}.$$

□

Corollary 1. *There exists an irrational number.*

Activity:

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1. Modify the argument in **Theorem 2** to show that there exists a positive real number y such that $y^2 = 3$.