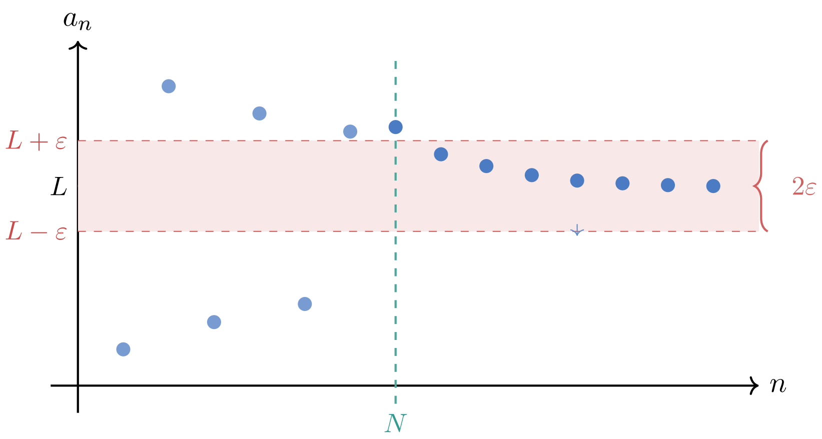
Limit of functions

From Sequential to Functional Limits

We have a precise definition of $\lim_{n\to\infty} a_n = L$ for a sequence. The idea behind $\lim_{x\to c} f(x) = L$ is analogous: $f(x)$ gets arbitrarily close to L whenever x is chosen close enough (but not equal) to c.

Recall the ε-N definition of limit: for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that whenever $n \geq N$,

$$|a_n - L| < \varepsilon.$$



Rewrite the definition of $a_n \to L$ from the picture above using ε neighborhood of L:

For functional limits, the challenge is still an ε-neighborhood around L, but the response is a δ-neighborhood around c.

The ε–δ Definition

Definition 1. Let $f : A \to \mathbb{R}$, and let c be a *limit point* of the domain A. We say

$$\lim_{x\to c} f(x) = L$$

provided that, for all $\varepsilon > 0$, there exists $\delta > 0$ such that whenever

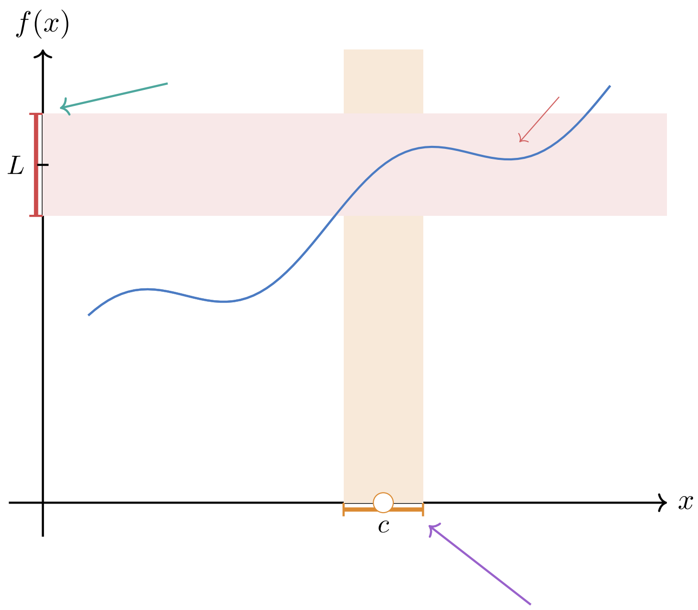
$$0 < |x - c| < \delta \quad (\text{and } x \in A) \qquad \text{it follows that} \qquad |f(x) - L| < \varepsilon.$$

Important remarks:

(i) The condition $0 < |x - c|$ means $x \neq c$: *what happens at c itself is irrelevant.*

(ii) The point c need ________ even be in the domain of f.

(iii) We only discuss $\lim_{x\to c} f(x)$ when c is a ________________ of A (so c can be approached).



Reading the picture: The challenge is any $\varepsilon > 0$ (setting the height of the red horizontal band around L). The response is a $\delta > 0$ (setting the width of the orange vertical strip around c). The definition says: *the entire portion of the graph inside the orange strip (excluding $x = c$) must lie inside the red band.*

Topological Version

Definition 2. Let c be a limit point of the domain of $f : A \to \mathbb{R}$. We say $\lim_{x \to c} f(x) = L$ provided that, for every ε-neighborhood $V_\varepsilon(L)$ of L, there exists a δ-neighborhood $V_\delta(c)$ such that for all $x \in V_\delta(c)$ with $x \neq c$ (and $x \in A$):

$$f(x) \in V_\varepsilon(L).$$

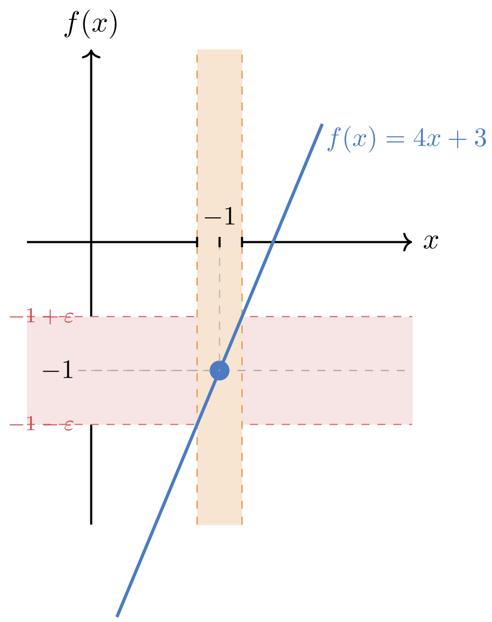
Examples

Example 1. Prove that $\displaystyle\lim_{x \to -1} (4x + 3) = -1$.

Scratch work:

. . . .

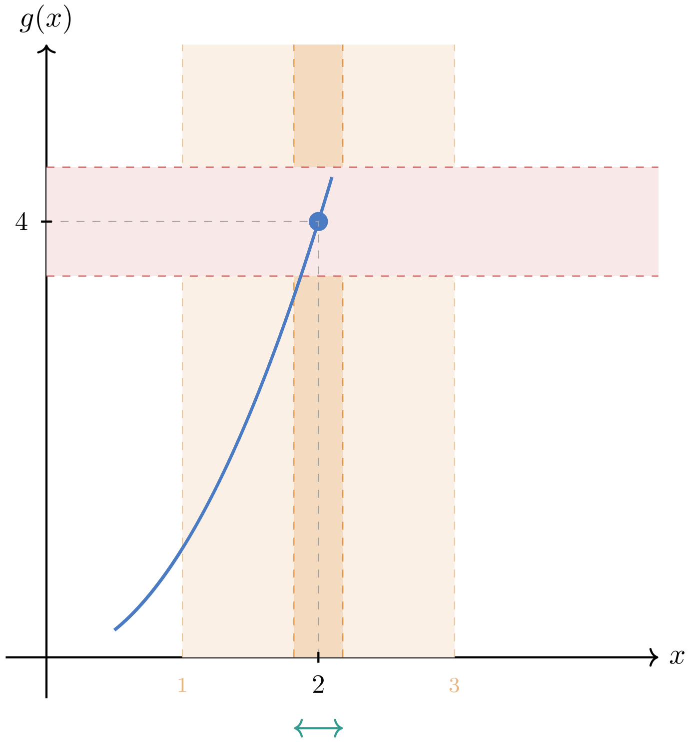
Formal proof:



Example 2. Prove $\lim_{x \to 2} x^2 = 4$.

Scratch work:

Formal proof:



Sequential Criterion for Functional Limits

Theorem 1. Given $f : A \to \mathbb{R}$ and a limit point c of A, the following are equivalent:

(i) $\lim_{x \to c} f(x) = L$.

(ii) For all sequences $(x_n) \subseteq A$ with $x_n \neq c$ and $(x_n) \to c$, we have $f(x_n) \to L$.

Proof sketch: $(i) \implies (ii)$**:** Assume $\lim_{x \to c} f(x) = L$. Let $(x_n) \to c$ with $x_n \neq c$.

Use the definition of $\lim_{x \to c} f(x) = L$ to show that $f(x_n) \to L$.

$(ii) \implies (i)$**:** By contrapositive: assume $\lim_{x \to c} f(x) \neq L$.

Write what it means to say that $\lim_{x \to c} f(x) \neq L$.

For each $n \in \mathbb{N}$, apply the negation above with $\delta = \frac{1}{n}$ to obtain x_n with

$$0 < |x_n - c| < \frac{1}{n} \quad \text{and} \quad |f(x_n) - L| \geq \varepsilon_0.$$

Explain why $(x_n) \to c$ but $f(x_n) \not\to L$, and why this contradicts (ii).

Divergence Criterion: Proving Limits Don't Exist

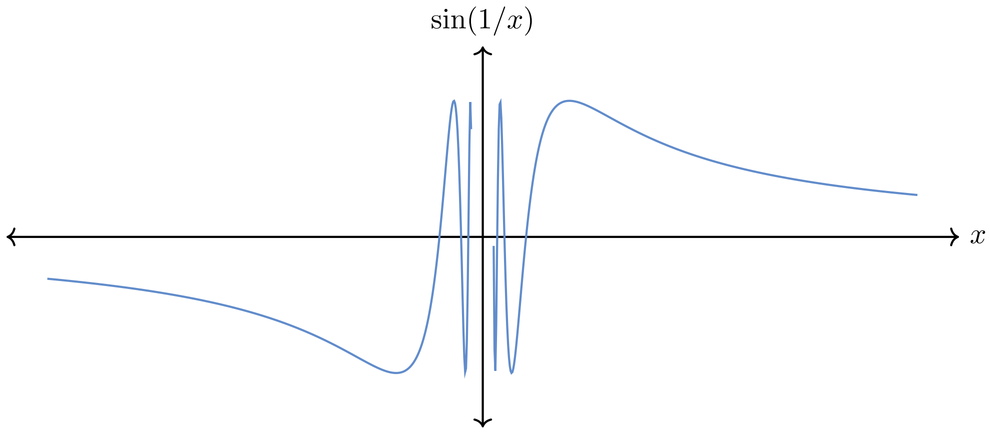
Corollary 1. Let $f : A \to \mathbb{R}$ and c a limit point of A. If there exist sequences (x_n) and (y_n) in A with $x_n \neq c$, $y_n \neq c$, and

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = c \qquad \text{but} \qquad \lim_{n \to \infty} f(x_n) \neq \lim_{n \to \infty} f(y_n),$$

then $\lim_{x \to c} f(x)$ **does not exist**.

Example 3. Show $\lim_{x\to 0} \sin(1/x)$ does not exist.

The function oscillates increasingly rapidly near 0*:*



Take $x_n =$ ________ and $y_n =$ ________.

Checklist:

1. Show that $x_n \to 0$ and $y_n \to 0$. ☐

2. Check $\sin(1/x_n) \neq \sin(1/y_n)$. ☐

Since $\lim \sin(1/x_n) = 0 \neq 1 = \lim \sin(1/y_n)$, the limit does not exist. ■

The Algebraic Limit Theorem for Functional Limits

Just as we built up arithmetic for sequences, the Sequential Criterion lets us inherit the results immediately:

Corollary 2 (Algebraic Limit Theorem). Assume $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$. Then:

(i) $\lim_{x\to c} kf(x) =$ __________ for all $k \in \mathbb{R}$,

(ii) $\lim_{x\to c}[f(x) + g(x)] =$ ____________________,

(iii) $\lim_{x\to c}[f(x)g(x)] =$ __________,

(iv) $\lim_{x\to c} \frac{f(x)}{g(x)} =$ __________, provided $M \neq 0$.

Activity

Problem 1. Prove that

$$\lim_{x \to 2} (3x + 4) = 10.$$

using Definition 1.

Problem 3. Decide whether each claim is **true** or **false**, and give a brief justification:

(a) Show that if $\delta > 0$ is a valid choice for a given $\varepsilon > 0$, then any δ' with $0 < \delta' < \delta$ is also a valid choice for the same ε.

(b) If $\lim_{x \to a} f(x) = L$ and $a \in \text{dom}(f)$, then $L = f(a)$.