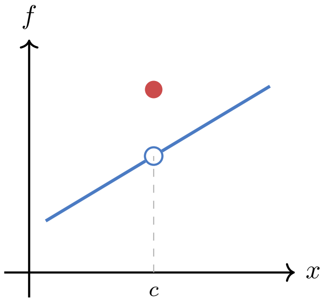
Continuous Functions

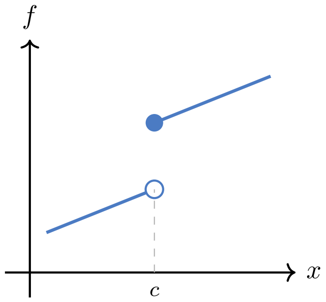
Motivation: Why Continuity Needs a Definition

You have been using the word *continuous* since precalculus: a function is continuous if you can draw its graph *without lifting your pen.* This picture is compelling, but it is not a definition – it is an intuition. And intuitions can fail.

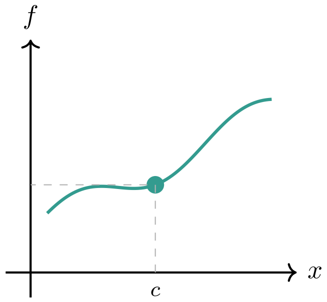
Consider the following:



Removable



Jump



Continuous

Both of the first two functions have a well-defined value at c, and both look "almost" continuous there. What precisely separates them from the third?

We now have a rigorous definition of $\lim_{x \to c} f(x)$ from the last handout. It is time to use it.

Definition of continuity

Definition 1. A function $f : A \to \mathbb{R}$ is **continuous at a point** $c \in A$ if, for all $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $|x - c| < \delta$ (and $x \in A$):

$$|f(x) - f(c)| < \varepsilon.$$

If f is continuous at every point in A, we say f is **continuous on** A.

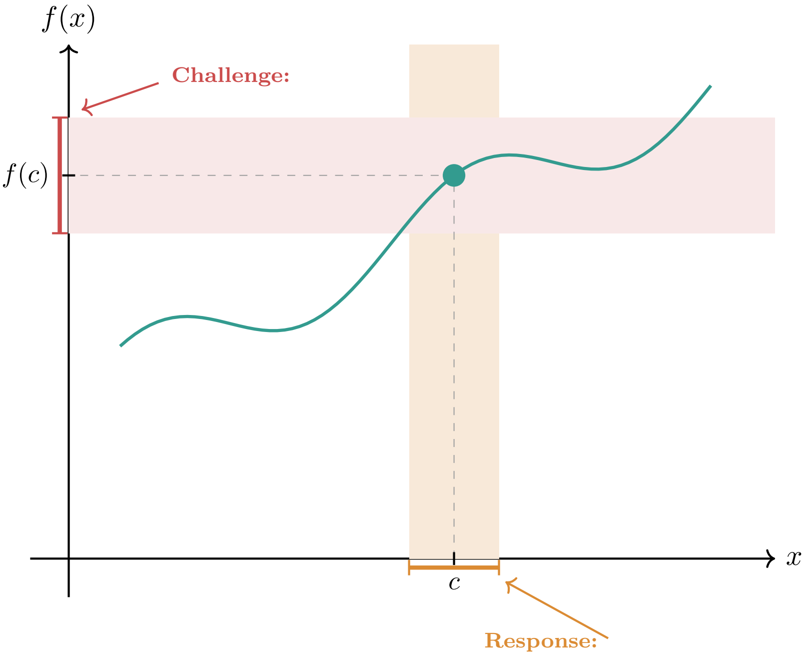
Takeaways: Continuity at c means three things simultaneously:

(i) $f(c)$ is defined (c is in the domain),

(ii) $\lim_{x \to c} f(x)$ exists,

(iii) $\lim_{x \to c} f(x) = f(c)$.

Any one of these failing means f is **discontinuous** at c.



Reading the picture: Same orange/red band structure as before. The one change: the point c is now *solid* on the curve and *included* in the orange strip. We must have $f(c)$ land inside the red band – not just nearby points.

Three equivalent characterizations

Just as functional limits had an ε–δ version and a sequential version, continuity has multiple equivalent formulations. Each is useful in different situations.

Theorem 1. Let $f : A \to \mathbb{R}$ and $c \in A$. The function f is continuous at c if and only if any one of the following holds:

(i) **(ε-δ definition):** For all $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $|x - c| < \delta$ (and $x \in A$):

$$|f(x) - f(c)| < \varepsilon.$$

(ii) **(Topological version):** For every ε-neighborhood $V_\varepsilon(f(c))$ of the output value $f(c)$, there exists a δ-neighborhood $V_\delta(c)$ of the input c such that for all $x \in V_\delta(c)$ (with $x \in A$):

$$f(x) \in V_\varepsilon(f(c)).$$

(iii) **(Sequential):** For all sequences $(x_n) \to c$ with $x_n \in A$, it follows that

$$f(x_n) \to f(c)$$

.

……

If c is a limit point of A, all three are also equivalent to:

(iv) $\lim_{x \to c} f(x) = f(c)$.

When to use each characterization.

Version	**Best used for . . .**
ε–δ	Direct proofs that a specific function IS continuous
Sequential (iii)	Proving discontinuity: exhibit one bad sequence $(x_n) \to c$ with $f(x_n) \not\to f(c)$
(iv)	Inheriting results from the last handout (Algebraic Limit Theorem, etc.)

Corollary 1 (Criterion for Discontinuity). Let $f : A \to \mathbb{R}$, $c \in A$ a limit point. If there exists a sequence $(x_n) \subseteq A$ with $(x_n) \to c$ but $f(x_n) \not\to f(c)$, then f is **not continuous** at c.

Examples:

Example 1. Prove that the function

$$f(x) = 3x - 1$$

is continuous at every $c \in \mathbb{R}$.

Scratch work:

. . . .

Formal proof:

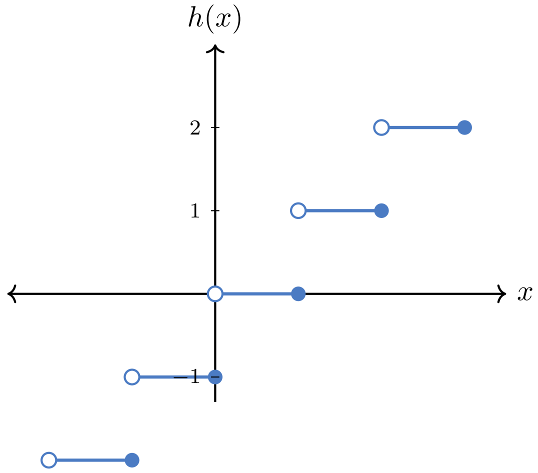
$\square$

Note: $\delta = \varepsilon/3$ works for *every* $c \in \mathbb{R}$: it does not depend on c at all. This will become important when we study *uniform continuity.*

Example 2 (Greatest Integer Function). Consider the function h defined by

$$h(x) = \lfloor x \rfloor.$$

Show h is discontinuous at each $m \in \mathbb{Z}$, but continuous at each $c \notin \mathbb{Z}$.



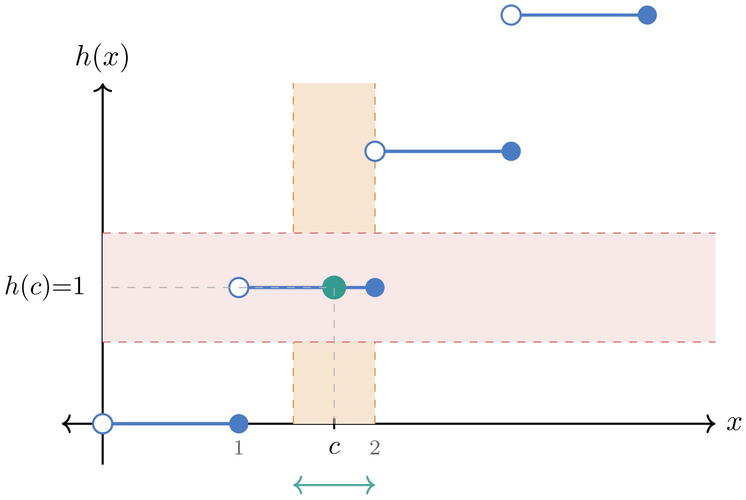
Discontinuity at integers. Let $m \in \mathbb{Z}$. We want to show $h = \lfloor x \rfloor$ is not continuous at m using the sequential criterion. We need to find a sequence $(x_n) \to m$ such that $h(x_n) \not\to h(m) = m$.

Continuity away from integers. Let $c \notin \mathbb{Z}$, so $n < c < n + 1$ for some $n \in \mathbb{Z}$. Choose:

$$\delta = \rule{4.5cm}{0.4pt}.$$

Explain why $|x - c| < \delta$ forces $h(x) = h(c)$ and hence $|h(x) - h(c)| < \varepsilon$ for any $\varepsilon > 0$.

Observation: Here δ does *not* depend on ε at all – the same δ works for every $\varepsilon > 0$.



Example 3. Prove that the function

$$f(x) = \sqrt{x}$$

is continuous on $A = [0, \infty)$.

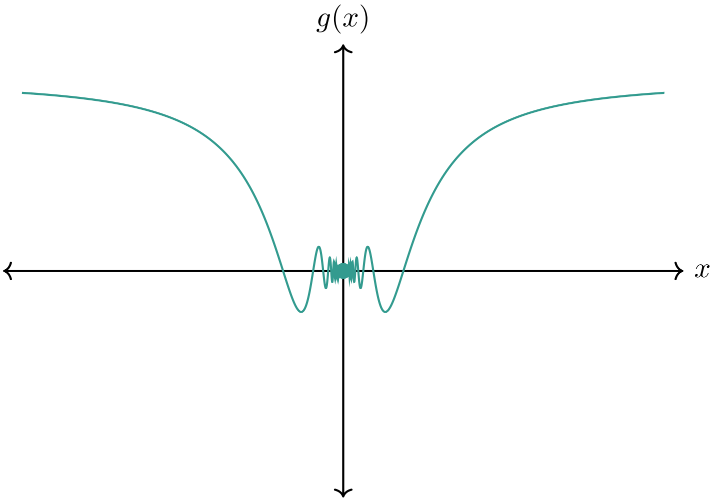
Case 1: $c = 0$.

Case 2: $c > 0$.
Scratch work:

Using the bound above, what should δ be? Then write the formal proof.

Example 4. Consider $g(x) = \begin{cases} x\sin(1/x) & x \neq 0 \\ 0 & x = 0. \end{cases}$

We saw that $\lim_{x\to 0} \sin(1/x)$ does not exist (Example 3 from the previous handout). But multiplying by x *kills* the oscillation. Is g continuous at 0?



Solution:

The Algebraic Continuity Theorem

We do not need to prove continuity from scratch for every function. The Sequential Criterion lets us inherit the Algebraic Limit Theorem:

Theorem 2. Assume $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ are both continuous at $c \in A$. Then:

(i) $kf(x)$ is continuous at c for all $k \in \mathbb{R}$,

(ii) $f(x) \pm g(x)$ is continuous at c,

(iii) $f(x) \cdot g(x)$ is continuous at c,

(iv) $\dfrac{f(x)}{g(x)}$ is continuous at c, provided $g(c) \neq 0$.

Consequence: All polynomials are continuous on $\mathbb{R}$.

Why? Start from two facts:

(a) The constant function $f(x) = k$ is continuous everywhere ($\delta = 1$ works for any $\varepsilon > 0$).

(b) The identity function $g(x) = x$ is continuous everywhere ($\delta = \varepsilon$ works).

By the Algebraic Continuity Theorem (repeatedly applying parts (i)–(iii)):

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

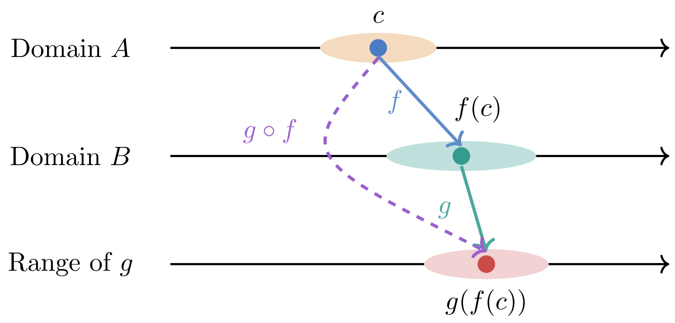
is a sum and product of continuous functions, hence continuous on $\mathbb{R}$.

By part (iv): **rational functions** are continuous wherever the denominator is nonzero.

Continuity of Compositions

The Algebraic Continuity Theorem handles sums and products. But what about $\sqrt{3x^2+5}$? We need to compose two continuous functions. The Algebraic Continuity Theorem does *not* cover this.

Theorem 3. Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ with $f(A) \subseteq B$. If f is continuous at $c \in A$ and g is continuous at $f(c) \in B$, then $g \circ f$ is continuous at c.



Proof sketch. Let $(x_n) \to c$ with $x_n \in A$.

Since f is continuous at c: $f(x_n) \to$ ___________.

Since g is continuous at $f(c)$ and $f(x_n) \to f(c)$: $g(f(x_n)) \to$ ___________.

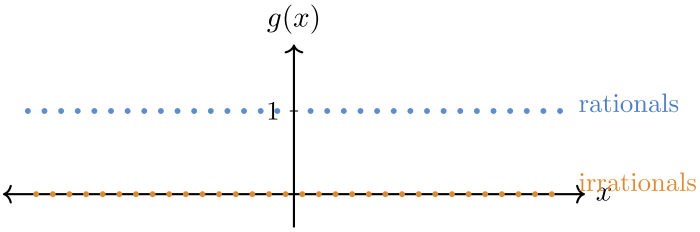
That is: $(g \circ f)(x_n) \to g(f(c)) = (g \circ f)(c)$. By the sequential criterion, $g \circ f$ is continuous at c. ☐

Example 5. $h(x) = \sqrt{3x^2+5}$ is continuous on all of $\mathbb{R}$.

Dirichlet's Function:

Define a function $g : \mathbb{R} \to \mathbb{R}$ as follows:

$$g(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q}. \end{cases}$$



Fact. Dirichlet's function g is **nowhere continuous** on $\mathbb{R}$: it is discontinuous at every single point of the real line.

One small but useful observation:

Fact. If c is an *isolated point* of A, then f is automatically continuous at c, for any f.

Why?

Activity

Problem 1. Use Definition 1 (the ε–δ definition of continuity) to prove that $f(x) = x^2 + 1$ is continuous at $c = 2$.

Problem 3. True/False:
If f and g are both discontinuous at c, then $f + g$ is discontinuous at c.