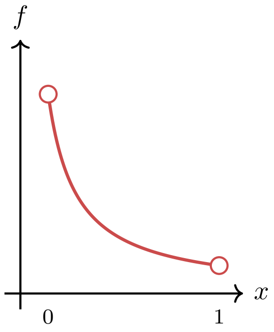
Continuous functions on compact sets and Uniform continuity

Warm-up:

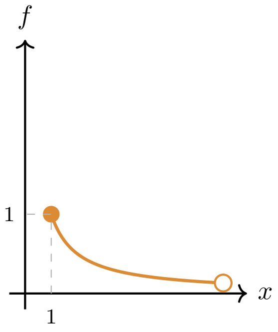
Consider the function

$$f(x) = \frac{1}{x}.$$

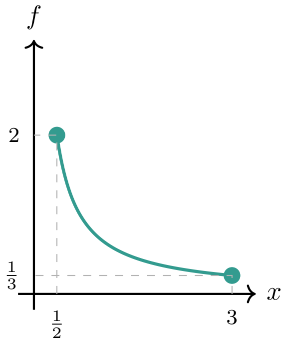
We will study its behavior on three different domains. The function itself does not change. Only the domain does.



$A_1 = (0,1)$



$A_2 = [1,\infty)$



$A_3 = [\frac{1}{2},3]$

Answer each question for each of the three domains $A_1 = (0,1)$, $A_2 = [1,\infty)$, $A_3 = [\frac{1}{2},3]$.

(a) Is $f(x) = 1/x$ continuous on the given domain?

(b) What is the range $f(A_i)$ in each case? Is the range open, closed, bounded, compact?

Domain	**Range** $f(A_i)$	**Open?**	**Closed?**	**Bounded?**	**Compact?**
$A_1 = (0,1)$					
$A_2 = [1,\infty)$					
$A_3 = [\frac{1}{2},3]$					

(c) On which domain(s) does f attain a maximum value? A minimum value?

(d) Recall the ε–δ definition of continuity. For a fixed $\varepsilon = \frac{1}{2}$, can you find a single $\delta > 0$ that works simultaneously for *every* point in A_1? What about in A_3? What is going wrong near $x = 0$ in A_1?

(e) What property does A_3 have that A_1 and A_2 lack? State it precisely using vocabulary from Chapter 3.

The pattern. The answers to (b)–(d) are perfect on A_3 and fail on A_1 and A_2 – and the only structural difference is __________________. This is not a coincidence. The rest of this handout explains exactly why __________________ forces these good behaviors, and proves it rigorously.

Preservation of Compact Sets

Theorem 1. Let $f : A \to \mathbb{R}$ be continuous on A. If $K \subseteq A$ is compact, then $f(K)$ is compact as well.

Proof. Let $K \subseteq A$ be a compact set and (y_n) be an arbitrary sequence in $f(K)$.

Use compactness of K and continuity of f to show that the sequence (y_n) has a convergent subsequence (y_{n_k}) converges to a limit in $f(K)$.

The Extreme Value Theorem

Theorem 2 (Extreme Value Theorem)**.** If $f : K \to \mathbb{R}$ is continuous on a compact set $K \subseteq \mathbb{R}$, then f attains both a maximum and minimum value. That is, there exist $x_0, x_1 \in K$ such that

$$f(x_0) \leq f(x) \leq f(x_1) \quad \text{for all } x \in K.$$

Why each hypothesis is necessary:

Condition dropped	**Example**	**What fails**
f not continuous		
K not closed		
K not bounded		

Example 1. Let $f(x) = \cos(x)$ defined on $K = [0, \pi]$. Without computing, explain why f attains both a maximum and minimum, and identify what they are.

Uniform Continuity

Recall the definition of ordinary continuity: f is continuous at c if, given $\varepsilon > 0$, there exists $\delta > 0$ (possibly depending on c) such that $|x - c| < \delta$ implies $|f(x) - f(c)| < \varepsilon$.

The subtlety is the phrase "possibly depending on c." Uniform continuity removes this dependence entirely.

Definition 1 (Uniform Continuity)**.** A function $f : A \to \mathbb{R}$ is uniformly continuous on A if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x, y \in A$:

$$|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

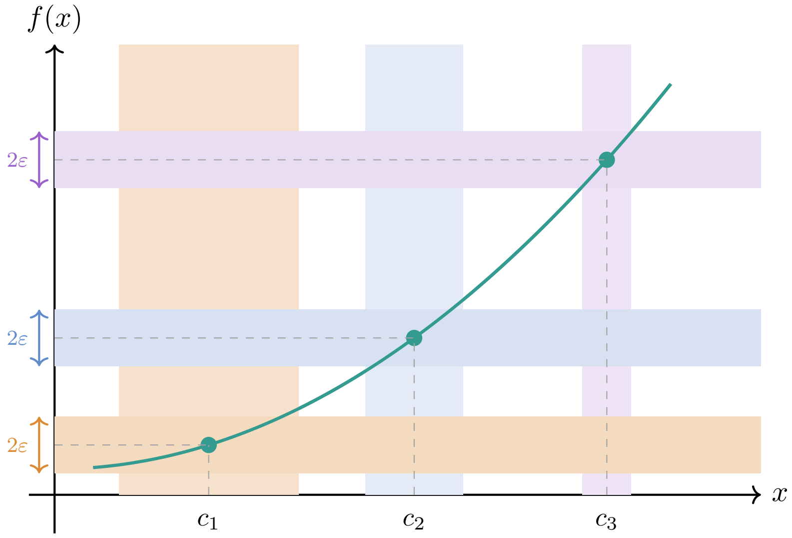
Comparison. In standard continuity, the quantifier order is:

$$\forall \varepsilon > 0,\ \forall c \in A,\ \exists \delta > 0\ \text{(may depend on } \varepsilon \textit{ and } c\text{)} : \quad |x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon.$$

In uniform continuity, the quantifier order is:

$$\forall \varepsilon > 0,\ \exists \delta > 0\ \text{(depends on } \varepsilon \textit{ only}\text{)} : \quad \forall x, y \in A,\ |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

The δ must work simultaneously for all pairs x, y in A.



Example 3: Prove that the function

$$f(x) = 2x + 3$$

is uniformly continuous on $\mathbb{R}$.

Scratch work:

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Formal proof:

Example 4: Prove that the function

$$f(x) = x^2$$

is uniformly continuous on $[0, 1]$.

Scratch work:

Formal proof:

Failure of Uniform Continuity:

Theorem 3. A function $f : A \to \mathbb{R}$ **fails** to be uniformly continuous on A if and only if there exists $\varepsilon_0 > 0$ and sequences (x_n), (y_n) in A satisfying

$$|x_n - y_n| \to 0 \quad \text{but} \quad |f(x_n) - f(y_n)| \geq \varepsilon_0.$$

Proof sketch. ($\Longrightarrow$) Assume $f : A \to \mathbb{R}$ is not uniformly continuous on A.

Write the negation of the definition of uniform continuity (Definition. 1)

……………………………………………………………………………………

Using the negation above, construct two sequences (x_n) and (y_n) in A such that $|x_n - y_n| \to 0$ and $|f(x_n) - f(y_n)| \geq \varepsilon_0$.

($\Longleftarrow$)

Example 4. Show that $f(x) = x^3$ is not uniformly continuous on $\mathbb{R}$.

Strategy: Find sequences (x_n) and (y_n) in $\mathbb{R}$ with $|x_n - y_n| \to 0$ but $|f(x_n) - f(y_n)| \not\to 0$.

Expand and estimate $\left(n + \frac{1}{n^2}\right)^3$ to show $|f(x_n) - f(y_n)| \to 3$ (you may use $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$).

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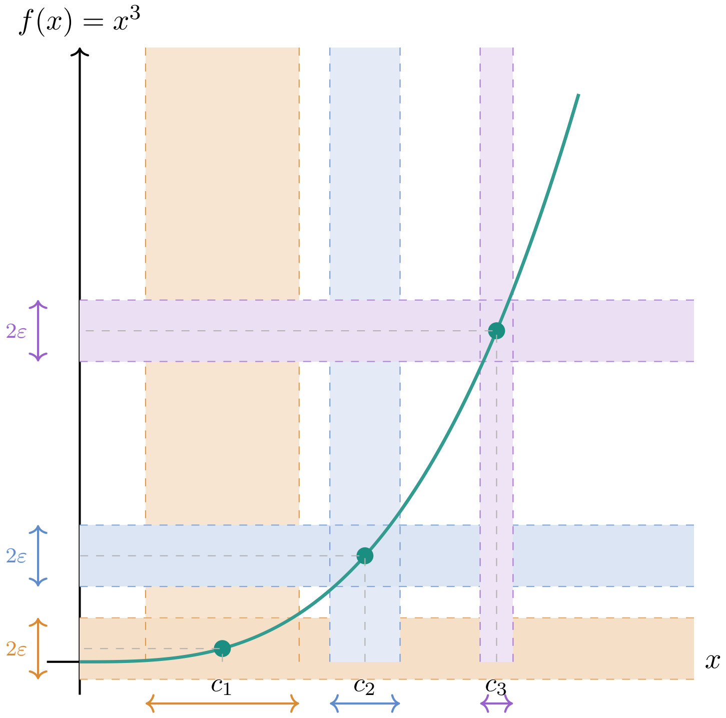
Conclude: $f(x) = x^3$ is not uniformly continuous on $\mathbb{R}$.



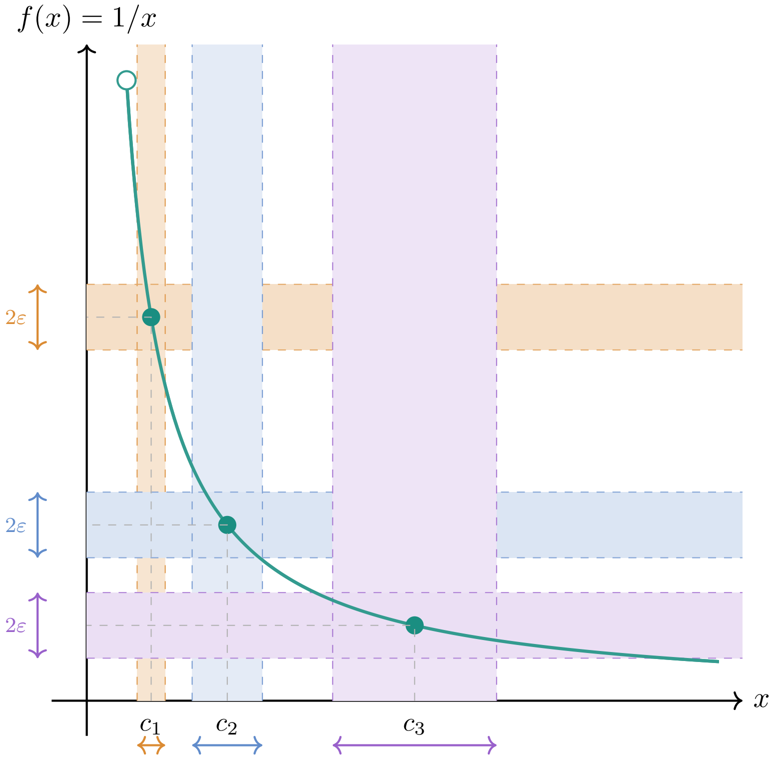
Example 5. Show that $h(x) = \dfrac{1}{x}$ is not uniformly continuous on $(0, 1)$.

Strategy: The problem is near $x = 0$ where $1/x$ blows up. Let $x_n =$ ______ and $y_n =$ ______.

(a) Verify that $|x_n - y_n| \to 0$.

(b) Compute $|h(x_n) - h(y_n)|$ and show it does not go to 0.

(c) Conclude h is not uniformly continuous on $(0, 1)$.



Uniform Continuity on Compact Sets

The previous two examples show that uniform continuity can fail on unbounded domains and on non-closed bounded domains. The next theorem closes the door: neither failure is possible when the domain is compact.

Theorem 4. If $f : K \to \mathbb{R}$ is continuous on a compact set $K \subseteq \mathbb{R}$, then f is uniformly continuous on K.

Proof. We argue by contradiction. Suppose f is continuous on K but not uniformly continuous. By Theorem 3, there exist $\varepsilon_0 > 0$ and sequences $(x_n), (y_n) \subseteq K$ with

$$|x_n - y_n| \to 0 \quad \text{but} \quad |f(x_n) - f(y_n)| \geq \varepsilon_0.$$

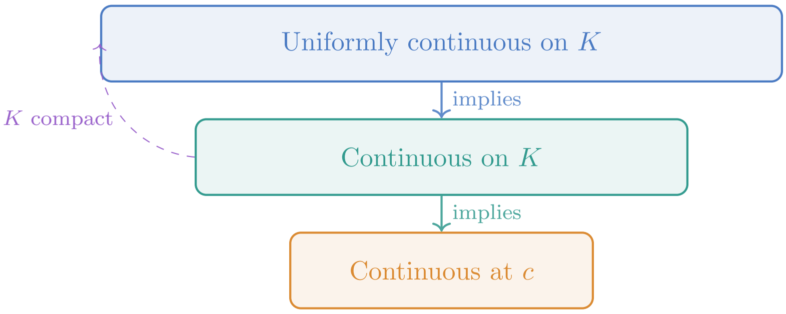
Use compactness of K to extract a convergent subsequence (x_{n_k}) of (x_n).

Now consider the corresponding subsequence (y_{n_k}) and show that (y_{n_k}) converges to the same limit as (x_{n_k}) does.

Use the continuity of f to show that $|f(x_{n_k}) - f(y_{n_k})| \to 0$.

But this contradicts $|f(x_n) - f(y_n)| \geq \varepsilon_0$ for all n. $\square$

Summary:



Activity

Problem 1. Prove using Definition 1 that $f(x) = x^3$ is uniformly continuous on $[-2, 2]$.

Problem 2. Is $f(x) = \dfrac{1}{x}$ uniformly continuous on $\left[\dfrac{1}{4}, 4\right]$?

Problem 3. True/False (justify each):

(a) If f is uniformly continuous on $(0, 1)$ and (x_n) is a Cauchy sequence in $(0, 1)$, then $(f(x_n))$ is also a Cauchy sequence.

(b) If f is continuous on $[0, \infty)$, then f is uniformly continuous on $[0, \infty)$.