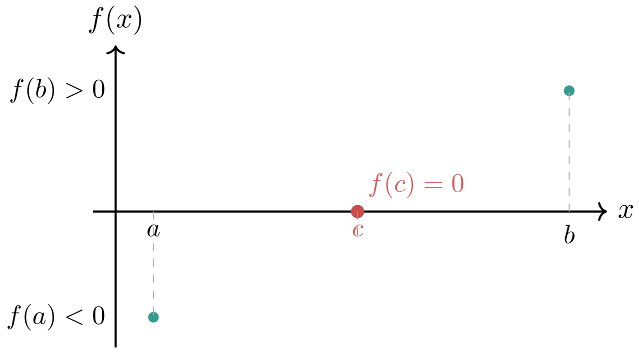
The Intermediate Value Theorem (IVT)

What does the IVT say?

Draw any continuous path from a point *below* the x-axis to a point *above* it, without lifting your pen. You *must* cross the axis somewhere.



The goal of this handout is to understand *why* it needs a proof, and *what* property of $\mathbb{R}$ makes it true.

Activity:

Q1. Consider $f(x) = x^2 - 2$ on $[1, 2]$. Compute $f(1)$ and $f(2)$. What does IVT (believe it for a moment) promise about $f(x) = 0$?

Q2. Now restrict entirely to $\mathbb{Q}$: work only with $x \in [1, 2] \cap \mathbb{Q}$. Does $f(x) = 0$ have a solution? Why not?

Q3. So f is continuous, $f(1) < 0 < f(2)$, yet the equation $f(x) = 0$ has no solution in $[1, 2] \cap \mathbb{Q}$. Which hypothesis of IVT is failing here?

Q4. What does $\mathbb{Q}$ lack that $\mathbb{R}$ has, and how does that gap allow f to "skip" the value 0?

Note. Every proof of IVT must use something $\mathbb{Q}$ lacks: __________________.

Warm-up: Does IVT apply?

For each scenario, decide (without computing anything) whether the IVT *guarantees* a solution. If yes, state what it guarantees. If no, explain which hypothesis fails.

(a) $f(x) = x^3 - 3x + 1$ on $[0, 1]$. Does $f(c) = 0$ for some $c \in (0, 1)$?

(b) $h(x) = x^2 - 2$ on $(1, 2)$. Does $h(c) = 0$ for some $c \in (1, 2)$?

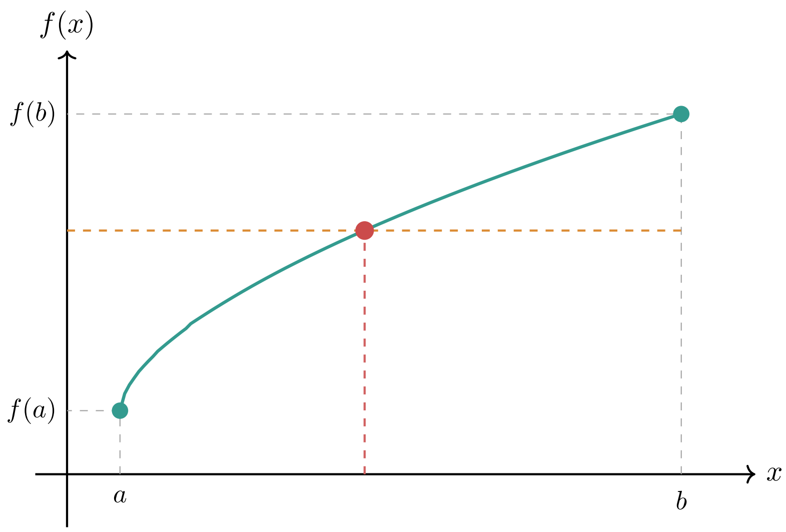
(c) $p(x) = \lfloor x \rfloor$ on $[0, 2]$. Does $p(c) = \frac{1}{2}$ for some $c \in (0, 2)$?

The Theorem

Theorem 1 (Intermediate Value Theorem)**.** Let $f : [a, b] \to \mathbb{R}$ be continuous. If L is a real number satisfying

$$f(a) < L < f(b) \qquad \text{or} \qquad f(a) > L > f(b),$$

then there exists a point $c \in (a, b)$ such that $f(c) = L$.

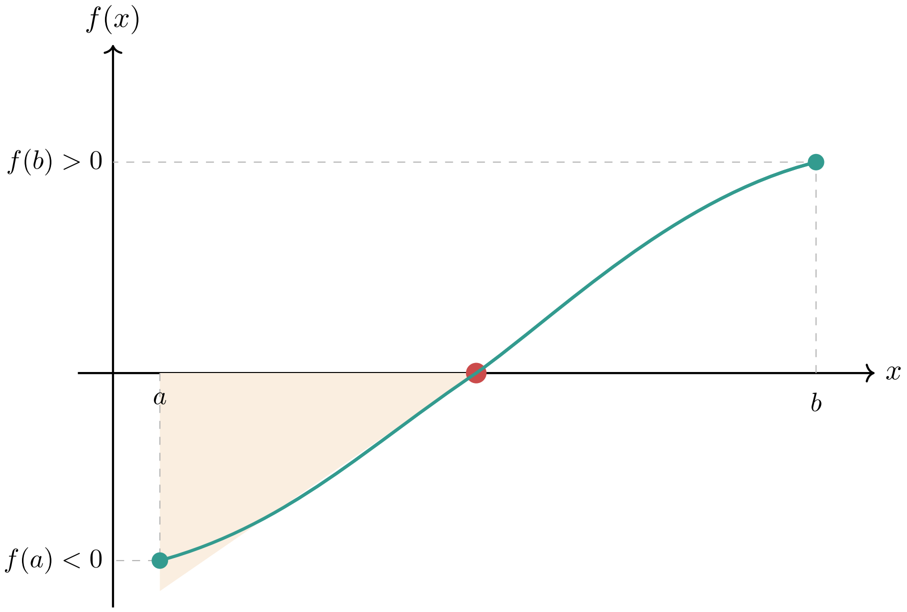


Why each hypothesis is necessary:

Condition dropped	Counterexample	What fails
f not continuous		
Domain not a closed interval (e.g. open)		
$\mathbb{R}$ replaced by $\mathbb{Q}$		

Proof via the Axiom of Completeness

We prove the special case $f(a) < 0 < f(b)$. The general case follows by replacing f with $f - L$.



Proof sketch. Let $K = \{x \in [a, b] : f(x) \leq 0\}$.

Step 1. Explain why K is non-empty and bounded above. Conclude $c = \sup K$ exists.

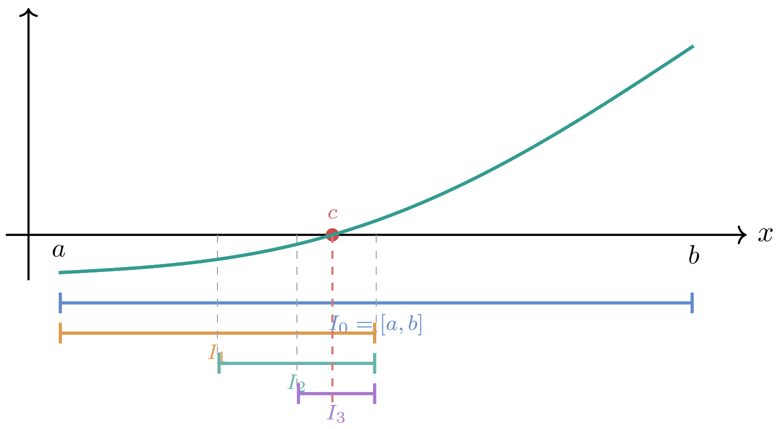
Step 2. Show $f(c) \leq 0$.

Step 3. Show $f(c) \geq 0$.

Step 4. Combine Steps 2 and 3. Conclude that $f(c) = 0$ and $c \in (a, b)$. □

Proof via the Nested Interval Property

The NIP gives an *algorithmic* proof: we *bisect* our way to the root.



Proof sketch. Again assume $f(a) < 0 < f(b)$. Let $I_0 = [a, b]$ and $z = \frac{a+b}{2}$.

Bisection rule:

$$I_1 = \begin{cases} \_\_\_\_\_\_\_\_\_\_ & \text{if } f(z) \geq 0 \\ \_\_\_\_\_\_\_\_\_\_ & \text{if } f(z) < 0. \end{cases}$$

In either case, $I_1 = [a_1, b_1]$ satisfies

- __________________.
- __________________.

Continuing inductively, we get nested closed intervals $I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$ with $|I_n| = \dfrac{b-a}{2^n}$.

(a) Invoke the NIP to find a point $c \in \displaystyle\bigcap_{n=0}^{\infty} I_n$.

What does the fact that $|I_n| \to 0$ tell you about a_n and b_n?

(b) By construction, $f(a_n) < 0$ and $f(b_n) \geq 0$ for all n. Conclude $f(c) \leq 0$ and $f(c) \geq 0$.

(c) Conclude $f(c) = 0$. □

Application: Existence of roots

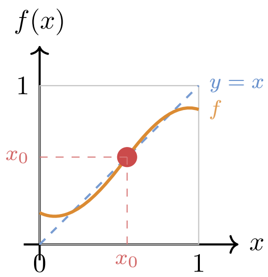
Example 1. Show that the polynomial $p(x) = x^5 - 3x^3 + x - 1$ has at least two real roots.

Strategy: Evaluate p at a few well-chosen integers and apply IVT on each sign-change interval.

Fixed Points

A fixed point of f is a point x_0 such that $f(x_0) = x_0$ – the function maps the point to itself.

Theorem 2. Let $f : [0, 1] \to [0, 1]$ be continuous. Then f has at least one fixed point, i.e., there exists $x_0 \in [0, 1]$ with $f(x_0) = x_0$.



Proof sketch. Define $g(x) = f(x) - x$ on $[0, 1]$.

(a) Compute $g(0)$ and $g(1)$ and find out the signs.

(b) Apply IVT to g to find x_0 with $g(x_0) = 0$, i.e., $f(x_0) = x_0$.

Preservation of connected sets

There is a more conceptual proof of IVT using the topology of $\mathbb{R}$.

Theorem 3. Let $f : G \to \mathbb{R}$ be continuous. If $E \subseteq G$ is connected, then $f(E)$ is connected.

Proof. Suppose for contradiction that $f(E)$ is not connected. Then there exist nonempty sets A, B with

________________, ________________, ________________.

Define the preimages $C = f^{-1}(A) \cap E$ and $D = f^{-1}(B) \cap E$.

(a) Verify that C and D are nonempty, and $E = C \cup D$.

(b) Show that $\overline{C} \cap D = \emptyset$ and $C \cap \overline{D} = \emptyset$, so that C and D are *separated.*

Corollary. IVT follows immediately.

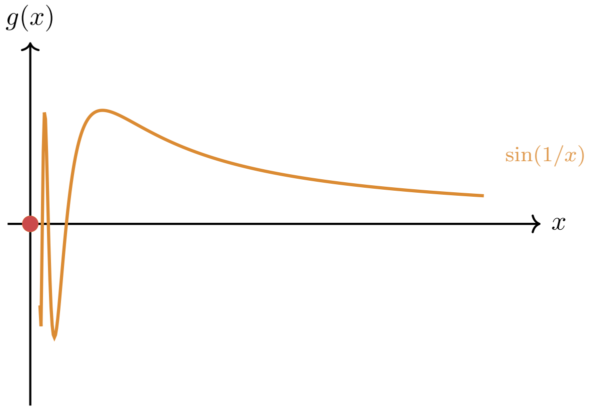
Does the converse hold?

Definition 1. A function f has the *intermediate value property (IVP)* on $[a, b]$ if for all $x < y$ in $[a, b]$ and all L between $f(x)$ and $f(y)$, there exists $c \in (x, y)$ with $f(c) = L$.

IVT says: continuous $\Rightarrow$ IVP. Is the converse true?

Counterexample. Consider

$$g(x) = \begin{cases} \sin\left(\dfrac{1}{x}\right) & x \neq 0 \\ 0 & x = 0. \end{cases}$$



(a) Is g continuous at 0?

(b) Despite this, it can be shown that g *does not* have the IVP on $[0, 1]$?

Activity

Problem 1. True/False (justify each):

(a) If $f : [a, b] \to \mathbb{R}$ is continuous and $f(a) \cdot f(b) < 0$, then f has at least one root in (a, b).

(c) If $f : [0, 1] \to [0, 1]$ is continuous, then f has exactly one fixed point.

(d) If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to+\infty} f(x) = +\infty$, then f has at least one root.