Sequences of functions:

You already know what it means for a sequence of *real numbers* (a_n) to converge: for every $\varepsilon > 0$, all terms are eventually within ε of the limit L.

Now suppose instead of a single number at each stage, we have a *function* $f_n : A \to \mathbb{R}$. For each fixed $x \in A$, evaluating the sequence at x gives a sequence of real numbers:

$$f_1(x),\ f_2(x),\ f_3(x),\ \dots,\ f_n(x),\ \dots$$

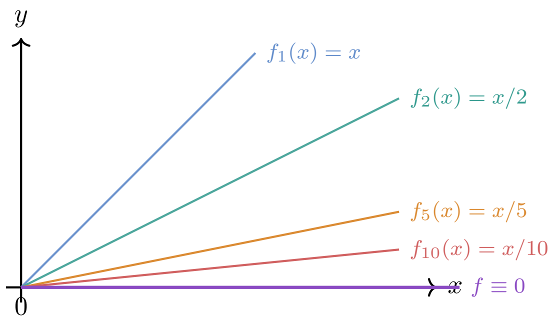
So (f_n) is simply a sequence whose terms happen to be functions rather than numbers. Let us consider the following examples:

$f_n(x)$	Domain	Write the first few terms of the sequence (f_n)
$\frac{x}{n}$	$\mathbb{R}$	
x^n	$[0,1]$	

Question: what does it mean for the sequence (f_n) to *converge*, and to what kind of object?

Activity: For each family below, fix an arbitrary x in the domain and ask: what does the sequence of *numbers* $f_1(x), f_2(x), f_3(x), \dots$ converge to? Compute at several specific values of x, then write a formula for the limit.

Family A. $f_n(x) = \dfrac{x}{n}$ on $\mathbb{R}$.

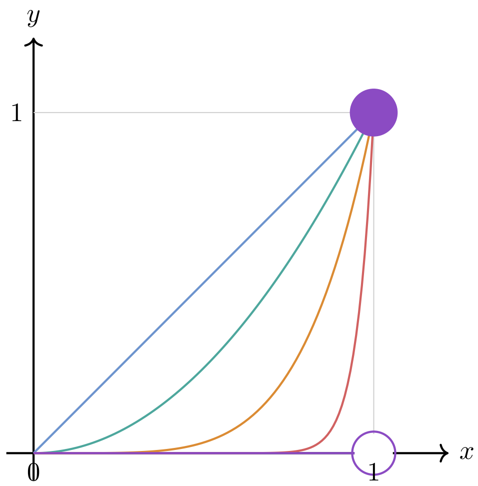


A1. Fix $x = 3$. Write out $f_1(3)$, $f_2(3)$, $f_5(3)$, $f_{10}(3)$. What do these numbers converge to?

A2. For an arbitrary fixed $x \in \mathbb{R}$, compute $\lim_{n\to\infty} \dfrac{x}{n}$.

A3. Write the limit function $f(x) = \lim_{n\to\infty} f_n(x)$. Is f continuous?

Family B. $g_n(x) = x^n$ on $[0, 1]$.



B1. Fix $x = 1/2$. Write out $g_1(1/2)$, $g_2(1/2)$, $g_5(1/2)$, $g_{10}(1/2)$. What do these numbers converge to?

B2. Fix $x = 9/10$. How large must n be before $g_n(9/10) < 0.01$? Compare this with the n needed for $x = 1/2$. What does the difference tell you?

B3. For arbitrary fixed $x \in [0, 1)$, compute $\lim_{n\to\infty} x^n$. What about $x = 1$? Write the limit function $g(x)$.

B4. Each g_n is continuous on $[0, 1]$. Is g continuous on $[0, 1]$?

What just happened?

In both families above, every f_n was *continuous*. Yet in each case the pointwise limit failed to be continuous at one point.

The obvious conjecture – the pointwise limit of continuous functions is continuous – is *false*.

The word "limit" is doing two different jobs simultaneously. Continuity of the limit requires *interchanging* the order of two limit operations:

$$\lim_{x\to c}\Big(\lim_{n\to\infty} f_n(x)\Big) \stackrel{?}{=} \lim_{n\to\infty}\Big(\lim_{x\to c} f_n(x)\Big).$$

This interchange is not free. The rest of this handout is about finding the right condition that makes it valid.

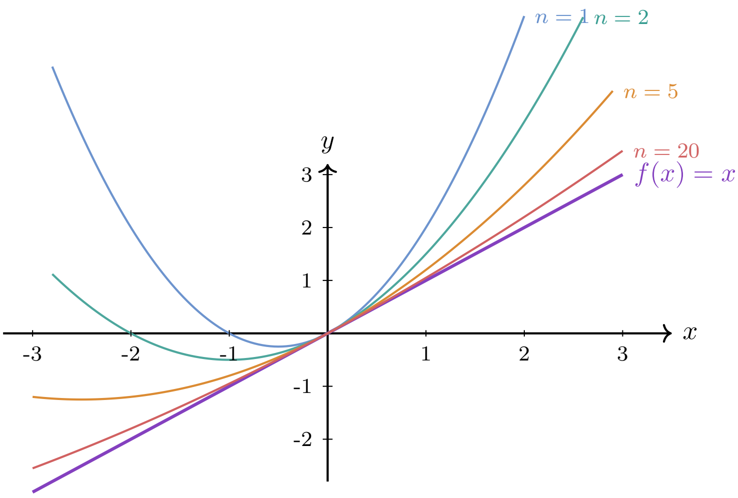
Pointwise Convergence

Definition 1 (Pointwise Convergence)**.** For each $n \in \mathbb{N}$, let f_n be a function defined on a set $A \subseteq \mathbb{R}$. The sequence (f_n) *converges pointwise on A to f* if, for every $x \in A$,

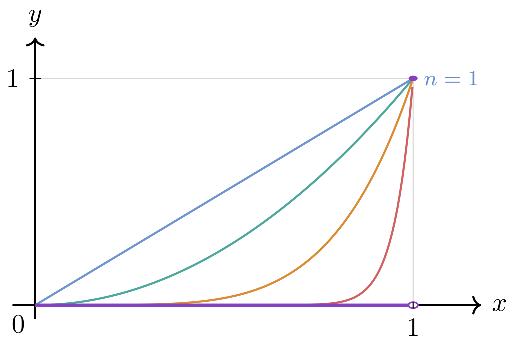
$$\lim_{n\to\infty} f_n(x) = f(x).$$

Equivalently: for every $\varepsilon > 0$ and every $x \in A$, there exists $N \in \mathbb{N}$ (which may depend on x) such that $n \geq N$ implies $|f_n(x) - f(x)| < \varepsilon$.

Example 1. Let $f_n(x) = \dfrac{x^2 + nx}{n}$ on $\mathbb{R}$. Show that $f_n \to f$ pointwise and identify f.



Example 2. Let $f_n(x) = x^n$ on $[0,1]$. Show that $f_n \to g$ pointwise and identify g.



Pointwise convergence is not enough:

Suppose (f_n) is a sequence of continuous functions on A converging *pointwise* to f. Attempt to prove f is continuous at $c \in A$.

Fix $c \in A$ and $\varepsilon > 0$. By the triangle inequality,

$$|f(x) - f(c)| \leq \underbrace{\qquad\qquad\qquad}_{\text{(I)}} + \underbrace{\qquad\qquad\qquad}_{\text{(II)}} + \underbrace{\qquad\qquad\qquad}_{\text{(III)}}.$$

(a) Term (III): Since $f_n(c) \to f(c)$, we can choose N so that term (III) $< \varepsilon/3$.

(b) Term (II): Once N is fixed, continuity of f_N provides $\delta > 0$ so term (II) $< \varepsilon/3$ whenever $|x - c| < \delta$.

(c) Why can't we control term (I) for all x with $|x - c| < \delta$ simultaneously?

Uniform Convergence

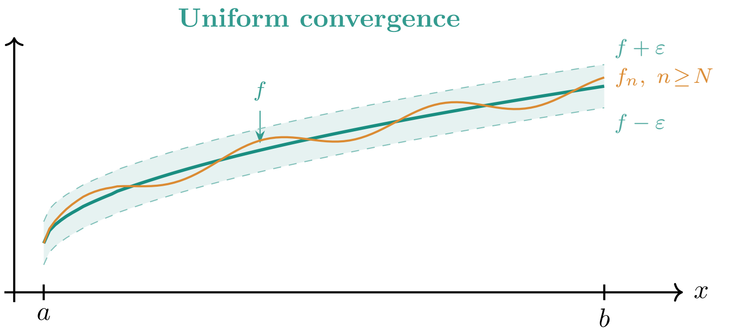
Definition 2 (Uniform Convergence)**.** The sequence (f_n) *converges uniformly on A to f* if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ (independent of x) such that

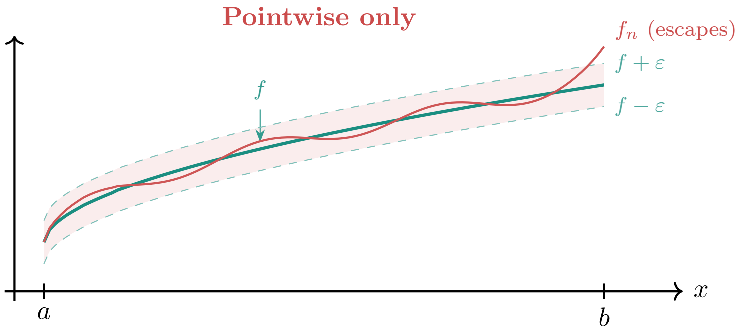
$$n \geq N \quad \Longrightarrow \quad |f_n(x) - f(x)| < \varepsilon \quad \text{for } \textit{all } x \in A.$$

The quantifier picture. The difference between the two definitions is entirely about *where the universal quantifier on x sits*:

Pointwise	$\forall \varepsilon > 0,\ \forall x \in A,\ \exists N \in \mathbb{N}$ (depending on both ε *and* x): $n \geq N \Rightarrow |f_n(x) - f(x)| < \varepsilon$
Uniform	$\forall \varepsilon > 0,\ \exists N \in \mathbb{N}$ (depending only on ε), $\forall x \in A$: $n \geq N \Rightarrow |f_n(x) - f(x)| < \varepsilon$

Geometric picture. Uniform convergence of $f_n \to f$ on A means: for every ε-band around the graph of f, there exists N such that the entire graph of f_n lies inside the band for all $n \geq N$.

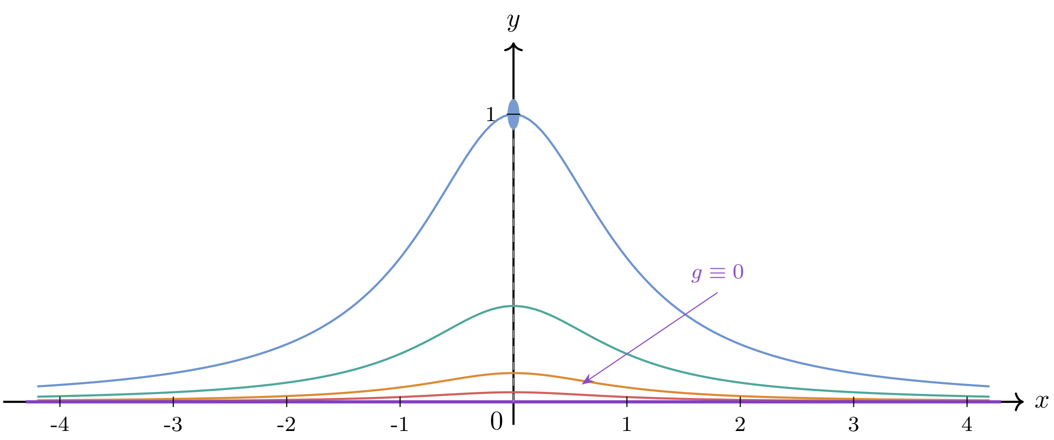




Example 3. Let $g_n(x) = \dfrac{1}{n(1+x^2)}$ on $\mathbb{R}$. Find the pointwise limit. Is convergence uniform?

Scratch work.

Formal proof.



Example 2. Let $f_n(x) = \dfrac{x^2 + nx}{n}$ on $\mathbb{R}$. Find the pointwise limit. Show convergence is *not* uniform on $\mathbb{R}$ but *is* uniform on every bounded interval $[-b, b]$.

Scratch work.

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Formal proof.

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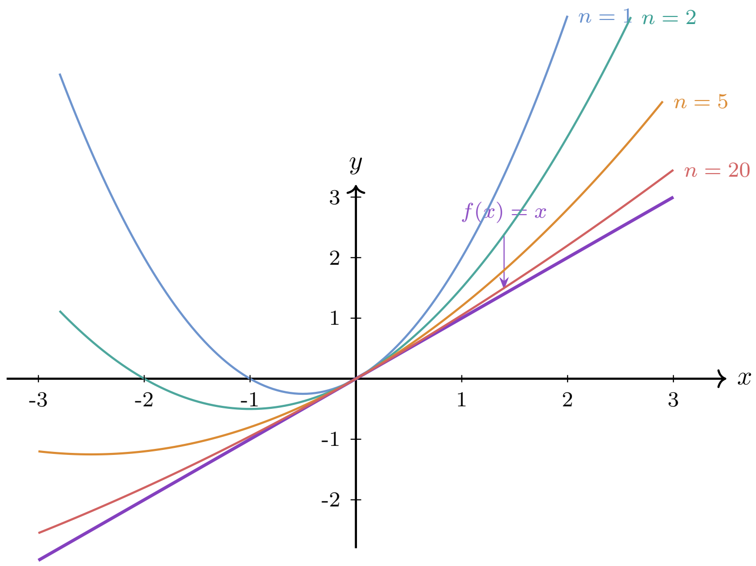

Cauchy criterion for uniform convergence

Theorem 1. A sequence of functions (f_n) defined on $A \subseteq \mathbb{R}$ converges uniformly on A if and only if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f_m(x)| < \varepsilon \qquad \text{whenever } m, n \geq N \text{ and } x \in A.$$

An important advantage of the Cauchy criterion is that it lets us verify uniform convergence *without knowing the limit function in advance.*

Proof sketch. ($\Rightarrow$) Suppose $f_n \to f$ uniformly.

– Write down the ε-N definition of uniform convergence for $f_n \to f$

– Use triangle inequality to show that for $m, n \geq N$ and $x \in A$:

$$|f_n(x) - f_m(x)| < \varepsilon.$$

($\Leftarrow$) Assume the Cauchy condition holds.

– Explain why for each fixed $x \in A$, the sequence $(f_n(x))$ is Cauchy in $\mathbb{R}$, and converges.

– Define $f(x) = \lim_{n\to\infty} f_n(x)$. Now show $f_n \to f$ uniformly:

Uniform convergence and Continuity:

Theorem 2. Let (f_n) be a sequence of functions defined on $A \subseteq \mathbb{R}$ converging uniformly on A to f. If each f_n is continuous at $c \in A$, then f is continuous at c.

Proof sketch: Fix $c \in A$ and $\varepsilon > 0$.

Step 1. Write the ε-N definition of $f_n \to f$ uniformly.

Step 2. Now $N \in \mathbb{N}$ is fixed. Write down the ε-δ definition of continuity for f_N at c.

Step 3. For $|x - c| < \delta$, apply the triangle inequality:

$$|f(x) - f(c)| \leq \_\_\_\_\_\_\_\_\_\_\_\_ + \_\_\_\_\_\_\_\_\_\_\_\_ + \_\_\_\_\_\_\_\_\_\_\_\_$$

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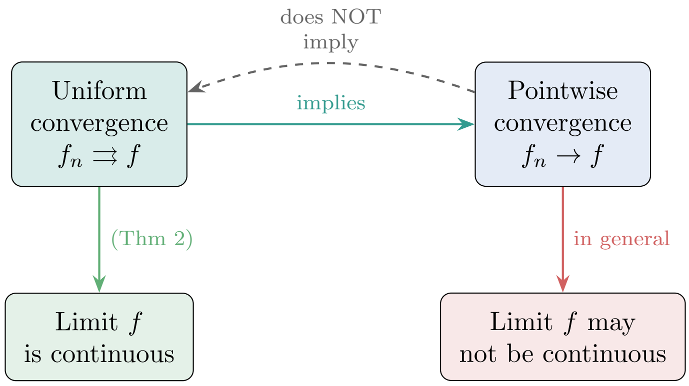
Since $\varepsilon > 0$ and $c \in \mathbb{R}$ was arbitrary, therefore, f is continuous at c. □

Summary Diagram



Activity

Problem 1. For each of the following, find the pointwise limit f, and determine whether convergence is uniform. Justify.

(a) $f_n(x) = \dfrac{nx}{1+nx^2} \quad$ on $(0, \infty)$.

(b) $f_n(x) = \dfrac{x}{1+nx^2}$ on $\mathbb{R}$.

Problem 2. For each statement, decide **True** or **False** and justify.

(a) If $f_n \to f$ uniformly on A and each f_n is bounded, then f is bounded.

(b) If $f_n \to f$ uniformly on A and each f_n is uniformly continuous on A, then f is uniformly continuous on A.