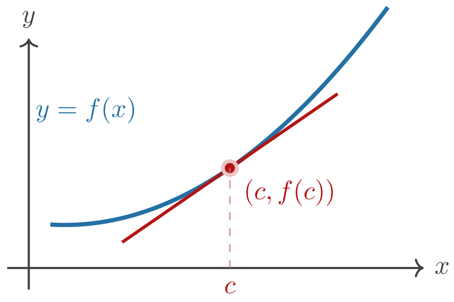
Uniform Convergence and Differentiation

Quick recap: the derivative.

We say $f : A \to \mathbb{R}$ is *differentiable at* $c \in A$ if the limit

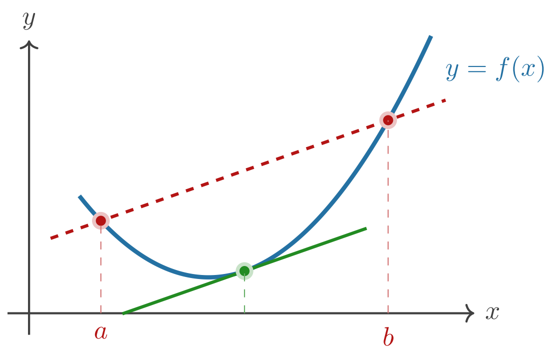
$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists. When $f'(c)$ exists, it equals the slope of the tangent line to the graph of f at c.



Two facts we will use:

- If f is differentiable at c, then f is continuous at c.
- (Mean Value Theorem) If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $\alpha \in (a, b)$ with $f'(\alpha) = \dfrac{f(b) - f(a)}{b - a}$.

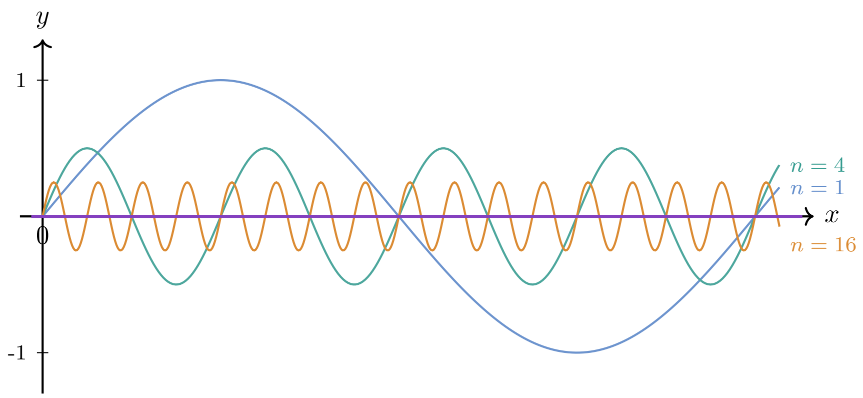


The question.

We proved: if $f_n \to f$ *uniformly* and each f_n is continuous, then f is continuous. The natural next question is:

If $f_n \to f$ uniformly and each f_n is differentiable, must f be differentiable?
And if so, does $f_n' \to f'$?

Example. Let $h_n(x) = \dfrac{\sin(nx)}{\sqrt{n}}$ on $\mathbb{R}$.



(a) Show that $h_n \to 0$ uniformly on $\mathbb{R}$.

(b) Compute $h_n'(x)$ for each n.

(c) Does (h_n') converge for any x?

Conclusion: $h_n \to 0$ uniformly, yet (h_n') *diverges everywhere.*

What is really being asked.

Differentiating the limit function means computing:

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{\left(\lim_{n \to \infty} f_n(x)\right) - \left(\lim_{n \to \infty} f_n(c)\right)}{x - c}.$$

Saying $f' = \lim_n f_n'$ is asking to interchange the following limits:

$$\lim_{x \to c} \lim_{n \to \infty} \frac{f_n(x) - f_n(c)}{x - c} \stackrel{?}{=} \lim_{n \to \infty} \lim_{x \to c} \frac{f_n(x) - f_n(c)}{x - c}.$$

This is even more delicate than the continuity interchange. The fix: require the *derivatives* (f_n') to converge uniformly.

The Differentiable Limit Theorem

Theorem 1. Let $f_n \to f$ pointwise on $[a, b]$, and assume each f_n is differentiable on $[a, b]$. If (f'_n) converges *uniformly* on $[a, b]$ to a function g, then f is differentiable and $f' = g$. That is,

$$\left(\lim_{n\to\infty} f_n\right)' = \lim_{n\to\infty} f'_n.$$

What the theorem says in plain language: To differentiate through a limit, you do not need (f_n) to converge uniformly – you need (f'_n) to converge uniformly. If the derivatives converge uniformly to some g, then the limit function f is differentiable and its derivative is exactly g.

Proof sketch. Fix $c \in [a, b]$ and $\varepsilon > 0$. We want to show $f'(c) = g(c)$, i.e., there exists $\delta > 0$ such that whenever $|x - c| < \delta$, it follows that

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By the triangle inequality, for any $n \in \mathbb{N}$ and $x \neq c$:

$$\left|\frac{f(x) - f(c)}{x - c} - g(c)\right| \leq \underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{\text{(I)}} + \underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{\text{(II)}} + \underbrace{\qquad\qquad\qquad}_{\text{(III)}}$$

Step 1. Use uniform convergence of (f'_n) to choose $N \in \mathbb{N}$ so that terms (I) and (III) are each $< \varepsilon/3$.

Step 2. How is the MVT used to control term (I)?

Step 3. With N fixed, use differentiability of f_N at c to choose $\delta > 0$ so term (II) $< \varepsilon/3$ whenever $|x - c| < \delta$.

Conclude: for $0 < |x - c| < \delta$, $\left| \frac{f(x) - f(c)}{x - c} - g(c) \right| <$ _____ + _____ + _____ = _____. □

Some notes about the proof.

1. Three-tool argument. Each tool below handles exactly one term – drop any one hypothesis and the proof collapses:

- Term (I) – MVT applied to $f_n - f_m$.
- Term (II) – differentiability of f_N at c.
- Term (III) – uniform convergence of (f_n') at the point c.

2. Order of quantifiers. Pick N *first*. Then, pick δ from the differentiability of the single function f_N. Reversing the order ties δ to a specific n and the argument breaks.

Examples

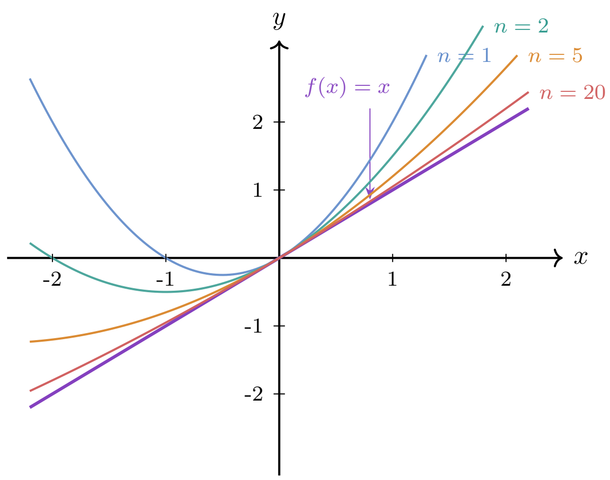
Example 1. Let $f_n(x) = \dfrac{x^2}{n} + x$ on $[-2, 2]$.

(a) Find the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$.

(b) Compute $f_n'(x)$ and find $g(x) = \lim_{n\to\infty} f_n'(x)$ pointwise.

(c) Show (f_n') converges *uniformly* to g on $[-2, 2]$.

(d) Conclude by Theorem 1 that $f' = g$. Verify directly that $f'(x) = g(x)$.



Example 2. Let $g_n(x) = \dfrac{x^n}{n}$ on $[0, 1]$.

(a) Show (g_n) converges uniformly on $[0, 1]$ and find $g = \lim_{n\to\infty} g_n$.

(b) Compute $g_n'(x)$ for each n. Show (g_n') converges pointwise on $[0, 1]$. Is the convergence uniform?

(c) Compute $\lim_{n\to\infty} g_n'(x)$ and compare it with $g'(x)$. Are they equal?

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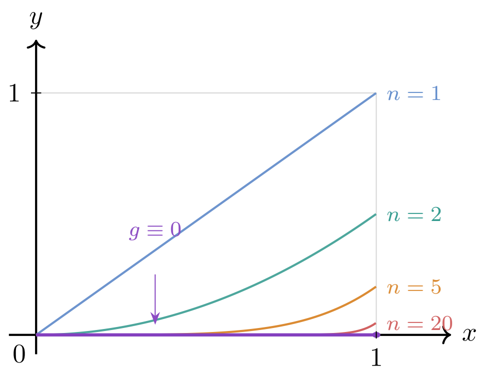

Note: This example shows that even when (f_n) converges uniformly, the sequence (f_n') may converge *non-uniformly*.

What hypothesis do we need?

Hypothesis	**Conclusion**	**Example**
$f_n \to f$ uniformly, each f_n differentiable	No conclusion about f'	$h_n(x) = \sin(nx)/\sqrt{n}$
$f_n \to f$ pointwise, $f_n' \rightrightarrows g$ uniformly	f is differentiable and $f' = g$	Example 1

Activity

Problem 1. Let $f_n(x) = \dfrac{nx + x^2}{2n}$ on $\mathbb{R}$.

(a) Compute $f = \lim_{n\to\infty} f_n$ pointwise. Identify f.

(b) Compute $f_n'(x)$ and find $\lim_{n\to\infty} f_n'(x)$ pointwise.

(c) Show (f_n') converges uniformly on every bounded interval $[-M, M]$.

(d) Apply the Theorem 1 to conclude $f' = \lim_{n\to\infty} f_n'$. Verify directly.

Problem 2. True/False – justify each.

(a) If $f_n \to f$ uniformly and each f_n is differentiable, then f is differentiable.

(b) If $f_n \to f$ pointwise, each f_n is differentiable, and $f_n' \to g$ pointwise, then $f' = g$.

(c) If $f_n \to f$ pointwise, each f_n is differentiable, and $f_n' \rightrightarrows g$ uniformly, then $f' = g$.