Sequences and Limits

In this handout, we develop a precise definition of what it means for a sequence to *converge to* a number and use it to distinguish between a sequence approaching a value and a sequence actually attaining that value.

Two Thought Experiments:

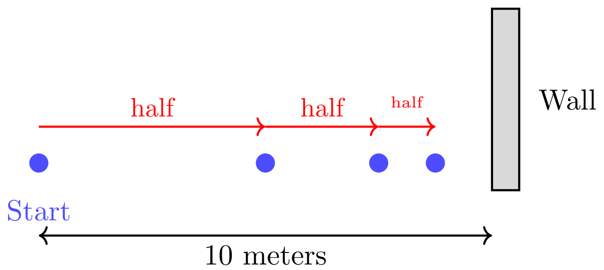
1. You are given two infinite lists of numbers. You are not told their formulas — only their first few terms.

Sequence A		**Sequence B**	
n	a_n	n	b_n
1	0.50	1	1
2	0.67	2	-1
3	0.75	3	1
4	0.80	4	-1
5	0.83	5	1
10	0.91	6	-1
50	0.98	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Question: What do you notice?

2. Imagine you're walking toward a wall...

You start 10 meters away from a wall. At each step, you walk *half of the remaining distance* to the wall.



Two Questions:

1. After many steps, is your distance from the wall getting *smaller*? ________________
2. Will you ever *reach* the wall in a finite number of steps? ______________________

Sequence: A sequence is a function

$$f\colon \mathbb{N} \to \mathbb{R}.$$

We write $f(n) = a_n$ and denote the sequence by __________.

Example: Common ways to describe sequences:

(i) Explicit list: $\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right)$

(ii) Formula: $(a_n)_{n=1}^{\infty}$ where $a_n =$ _______________.

(iii) Recursive: (x_n) where $x_1 = 2$ and $x_{n+1} = \frac{1}{2}(x_n + 1)$ for $n \geq 1$.

(iv) General term: $((-1)^n)_{n=1}^{\infty} = (-1, 1, -1, 1-, 1, 1, \ldots)$.

Note:

(i) A sequence is an _______________ list, not a set.

(ii) On occasion, it will be more convenient to index a sequence starting from $n = 0$ or $n = n_0$ for some $n_0 \in \mathbb{N}$.

Convergence of sequences:

Informally, a sequence (a_n) converges to a if: *after some point, all terms of the sequence stay close to a.*

Question: Can you explain why *Sequence A* "feels" convergent and *Sequence B* does not?

We need a precise definition:

Definition 1 (Convergence of a sequence). A sequence (a_n) converges to $a \in \mathbb{R}$ if, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that whenever __________, it follows that

____

In this case, we write

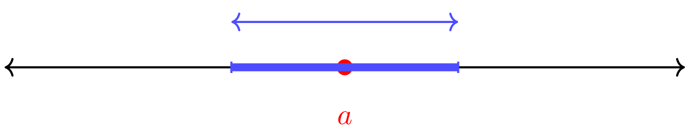
$$\lim_{n\to\infty} a_n = a, \quad \text{or} \quad a_n \to a.$$

Deciphering the definition:

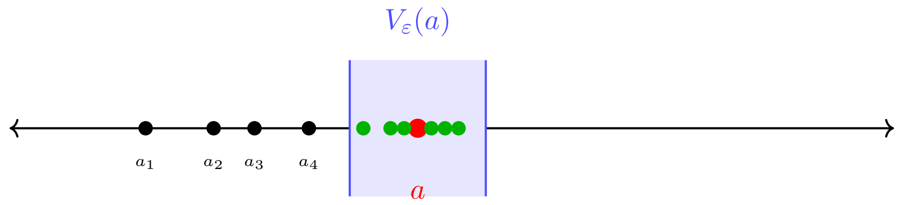
Definition 2 (ε-neighborhood). Given $a \in \mathbb{R}$ and $\varepsilon > 0$, the ε-neighborhood of a is

$$V_\varepsilon(a) = \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} = \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_}$$

Visually:



Geometric interpretation of $(a_n) \to a$:
$(a_n) \to a$ means: "Given *any* ε-neighborhood of a, eventually __________ terms of (a_n) are in that neighborhood."



Let's work with Examples:

Example: Prove that $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.
Strategy: Given ______, we need to find ______ such that whenever ________, it follows that

____

Scratch Work (DO THIS FIRST!):
We want: ___________.
Solving for n:

____

Key insight: If we choose N to be any integer greater than ___________, then $n \geq N$ will guarantee $\frac{1}{\sqrt{n}} < \varepsilon$!

Formal Proof. Let $\varepsilon > 0$ be arbitrary.

Choose $N \in \mathbb{N}$ such that ________________.
Now let $n \geq N$. Then:

Therefore, $\lim_{n\to\infty} \frac{1}{\sqrt{n}} =$ ______. □

Template for Convergence Proofs: To prove $\lim_{n\to\infty} a_n = a$:

1. **Start:** Let $\varepsilon > 0$.
2. **Scratch work:** Solve $|a_n - a| < \varepsilon$ for n.
3. **Choose N:** Choose N accordingly.
4. **Verify:** For $n \geq N$, verify $|a_n - a| < \varepsilon$.
5. **Conclude:** $\lim_{n\to\infty} a_n = a$.

Exercise 1: Prove that $\lim_{n\to\infty} \frac{n+1}{n} = 1$ using the definition of convergence of a sequence.

Divergence:

Definition 3. A sequence that ________________ converge is said to be a **divergent** sequence.

How to prove a sequence diverges?

Remember: $a_n \to a$ if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that whenever ___________, it follows that $|a_n - a| < \varepsilon$.

Negation of convergence:
$(a_n) \not\to a$ means: ________$\varepsilon > 0$ such that ________$N \in \mathbb{N}$, ________$n \geq N$ such that ___________________.

Example: Show that the sequence $(-1, 1, -1, 1, -1, 1, \dots)$ does not converge to 0.

Activity:

Exercise: Prove that

$$\lim_{n\to\infty}\left(\frac{2n+1}{5n+4}\right)=\frac{2}{5}$$

using the definition of a convergence of a sequence.