The Limit Theorems (Part I)

Having rigorously defined the convergence of sequences, we can now explore how limits interact with algebraic operations. This section establishes that the limiting process behaves predictably and intuitively with respect to arithmetic operations.

Why do we need these theorems?

We've defined convergence precisely, but computing limits using ε-N arguments for every sequence would be tedious. Consider:

$$\lim_{n\to\infty}\left(\frac{2n+1}{5n+4}\right)$$

Question: Must we use the definition directly, or can we manipulate limits algebraically?

For instance, could we say:

$$\lim_{n\to\infty}\left(\frac{2n+1}{5n+4}\right)=\frac{\lim_{n\to\infty}(2n+1)}{\lim_{n\to\infty}(5n+4)}\quad ?$$

Bounded Sequences:

Before establishing our main theorems, we need a preliminary result.

Definition 1. A sequence (x_n) is **bounded** if there exists a number $M > 0$ such that

____________________ for all $n \in \mathbb{N}$.

In other words, if a sequence (x_n) is bounded, then every term of the sequence lies in the interval $[-M, M]$.

Theorem 1. Every convergent sequence is _______________.

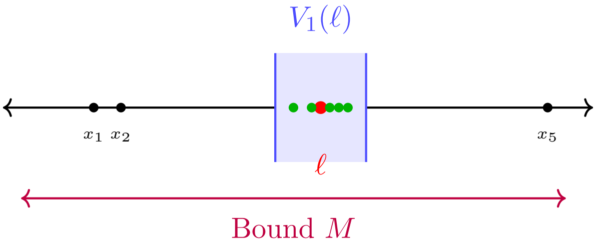
Proof: We will prove the theorem in steps.

Step 1: Suppose $(x_n) \to \ell$. Choose $\varepsilon = 1$. There exists $N \in \mathbb{N}$ such that whenever $n \geq N$, we have $|x_n - \ell| < 1$.

Step 2: For $n \geq N$: Show that $|x_n| < |\ell| + 1$.
Hint: Use the triangle inequality: $|x_n| = |x_n - \ell + \ell| \leq |x_n - \ell| + |\ell|$.

Step 3: For $n < N$: How many terms are there before the N-th term?
Answer: There are ______________ many terms: $x_1, x_2, \ldots, x_{N-1}$.

Step 4: Define $M = \max\{|x_1|, |x_2|, \ldots, |x_{N-1}|, |\ell| + 1\}$ and Show that $|x_n| \leq M$ for all $n \in \mathbb{N}$. **Hint:** Consider two cases: (i) $n < N$ and (ii) $n \geq N$.



Convergent sequences are eventually trapped in a bounded region, and there are only finitely many "wild" early terms.

☐

The Algebraic Limit Theorem

This fundamental theorem allows us to compute limits by breaking them into simpler pieces.

Theorem 2 (Algebraic Limit Theorem). Let $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Then:

(i) $\lim_{n\to\infty} (ca_n) =$ ____________________________ for all $c \in \mathbb{R}$;

(ii) $\lim_{n\to\infty} (a_n + b_n) =$ _____________________________________;

(iii) $\lim_{n\to\infty} (a_n b_n) =$ ____;

(iv) $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) =$ ________________________________, provided $b \neq 0$.

Proof of part (i): Let $c = 0$. Then the sequence (ca_n) reduces to $(0, 0, 0, 0, \ldots)$ and the theorem is true trivially.
Let $c \neq 0$. We want to show $(ca_n) \to ca$.
Scratch work:
Let $\varepsilon > 0$. We need to find __________ such that wherever __________, it follows that ____________________.

Now: $|ca_n - ca| =$ ____________________.
We know $(a_n) \to a$, so we can make $|a_n - a|$ as small as we like. Make:

$$\underline{\qquad\qquad\qquad\qquad}.$$

Formal proof:
Since $(a_n) \to a$, for the value __________, there exists $N \in \mathbb{N}$ such that whenever __________, it follows that

$$\underline{\qquad\qquad\qquad\qquad}$$

Therefore, for all $n \geq N$:

$$\begin{aligned} |ca_n - ca| &= \underline{\qquad\qquad\qquad\qquad} \\ &< \underline{\qquad\qquad\qquad\qquad} \\ &= \underline{\qquad}. \end{aligned}$$

Thus, $\lim_{n\to\infty}(ca_n) = ca$. ☐

Proof of part (ii): We want to show $(a_n + b_n) \to a + b$.

Let $\varepsilon > 0$. We want: $|(a_n + b_n) - (a + b)| < \varepsilon$.

Scratch work: Use the triangle inequality to split the problem.

$$\begin{aligned} |(a_n + b_n) - (a + b)| &= |(a_n - a) + (b_n - b)| \\ &\leq \underline{\qquad\qquad\qquad\qquad\qquad} \end{aligned}$$

The "$\varepsilon/2$ trick": Make each piece less than $\varepsilon/2$.

Formal proof: Let $\varepsilon > 0$ be arbitrary. Since $(a_n) \to a$, there exists $N_1 \in \mathbb{N}$ such that whenever $n \geq N_1$, it follows that

$$|a_n - a| < \underline{\qquad\quad}.$$

Since $(b_n) \to b$, there exists $N_2 \in \mathbb{N}$ such that whenever $n \geq N_2$, it follows that

$$|b_n - b| < \underline{\qquad\quad}.$$

Let $N =$ ____________________. Then for $n \geq N$, we have *both* inequalities:

$$\begin{aligned} |(a_n + b_n) - (a + b)| &\leq \underline{\qquad\qquad\qquad\qquad\qquad} \\ &< \underline{\qquad\qquad\qquad} \\ &= \underline{\qquad}. \end{aligned}$$

Therefore, $\lim\limits_{n\to\infty}(a_n + b_n) = a + b$. ☐

Proof of part (iv): To show $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right)=\frac{a}{b}$ (when $b\neq 0$), it suffices to prove:

$$\lim_{n\to\infty}\left(\frac{1}{b_n}\right)=\frac{1}{b},$$

then apply part (iii).

Challenge: We need to bound $\left|\frac{1}{b_n}-\frac{1}{b}\right|=\frac{|b-b_n|}{|b||b_n|}$.

Key steps:

1. Show $|b_n|$ is bounded *away from zero*: Find $N_1\in\mathbb{N}$ such that $n\geq N_1$ implies $|b_n|>\frac{|b|}{2}$.
   $\left(\textbf{Hint:}\text{ Use }\varepsilon_0=\frac{|b|}{2}\text{ in the definition of }(b_n)\to b.\right)$

2. Then $\frac{1}{|b||b_n|}<\frac{2}{|b|^2}$ for $n\geq N_1$.

3. Find $N_2\in\mathbb{N}$ such that $n\geq N_2$ implies $|b_n-b|<\frac{\varepsilon|b|^2}{2}$.

4. For $N=$ ________________ and $n\geq N$:

$$\begin{aligned}\left|\frac{1}{b_n}-\frac{1}{b}\right| &= \underline{\hspace{4cm}}\\ &< \underline{\hspace{4cm}}\\ &= \underline{\hspace{1.5cm}}.\end{aligned}$$

Example 1: Use the Algebraic Limit Theorem to compute $\lim_{n\to\infty}\left(\frac{2n+1}{5n+4}\right)$.

Solution:

Activity

Exercise 1: Complete the proof of part (iii): $\lim_{n\to\infty}(a_nb_n) = ab$.

Hints:

(a) Show that $|a_nb_n - ab| \le |b_n||a_n - a| + |a||b_n - b|$ by adding and subtracting ab_n.

(b) Use the fact that (b_n) is bounded: $|b_n| \le M$ for some $M > 0$.

(c) Make $|b_n||a_n - a| < \varepsilon/2$ and $|a||b_n - b| < \varepsilon/2$.