The Limit Theorems (Part II)

In the previous handout, we established the limits rules with respect to algebraic operations In this handout, we explore how limits interact with inequalities and ordering.

The Order Limit Theorem:

Limits also respect inequalities, with some important caveats.

Theorem 1. Assume $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Then:

(i) If $a_n \geq 0$ for all $n \in \mathbb{N}$, then ________________________________.

(ii) If $a_n \leq b_n$ for all $n \in \mathbb{N}$, then ________________________________.

(iii) If there exists $c \in \mathbb{R}$ with $c \leq b_n$ for all $n \in \mathbb{N}$, then ____________________________. Similarly, if $a_n \leq c$ for all $n \in \mathbb{N}$, then ____________________________________.

Observation: Strict inequalities in the sequence do NOT necessarily give strict inequalities in the limit.

Example: Consider $a_n = \frac{1}{n}$. We have $a_n >$ _____ for all $n \in \mathbb{N}$, but $\lim_{n\to\infty} a_n =$ __________.

So the limit is _______________ strict inequality, only ≥ 0.

Proof of part (i):
Suppose $a_n \geq 0$ for all n, and $(a_n) \to a$.

Assume for contradiction: _____________.

Consider $\varepsilon =$ __________. Since $(a_n) \to a$, there exists $N \in \mathbb{N}$ such that whenever $n \geq N$, it follows that

_______________________.

In particular, for $n = N$:

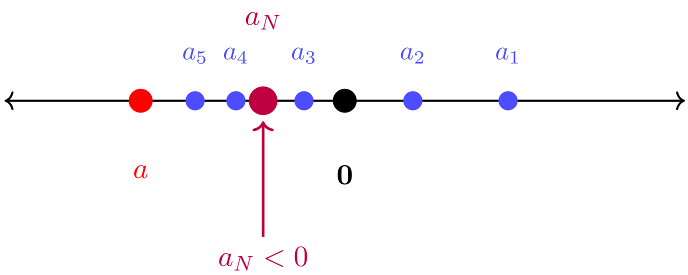
_______________________.

This means a_N lies in the interval ____________________________.

Since $a < 0$, we have:
$a - |a| =$ ____________________________ and $a + |a| =$ __________.

Therefore, a_N_____, which **contradicts** our hypothesis that $a_n \geq 0$.

Thus, ____________________. □



Proof of part (ii):
Assume $a_n \leq b_n$ for all $n \in \mathbb{N}$.

Then $b_n - a_n$ ______ 0 for all $n \in \mathbb{N}$.

By the Algebraic Limit Theorem:

$$\lim_{n\to\infty}(b_n - a_n) = \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n = \rule{3cm}{0.4pt}.$$

By part (i), since $b_n - a_n$ ______ 0 for all $n \in \mathbb{N}$, we have:

$$b - a \geq 0,$$

which gives $a \leq b$. □

The Squeeze Theorem

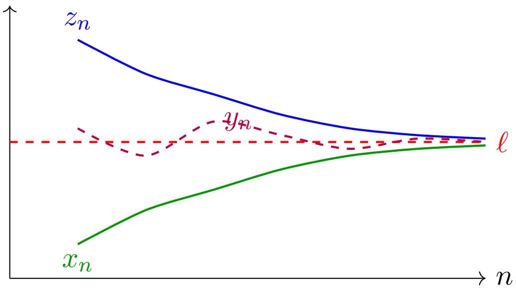
A powerful consequence of the Order Limit Theorem:

Theorem 2 (Squeeze Theorem). If $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$, and if

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = \ell,$$

then ________________________.

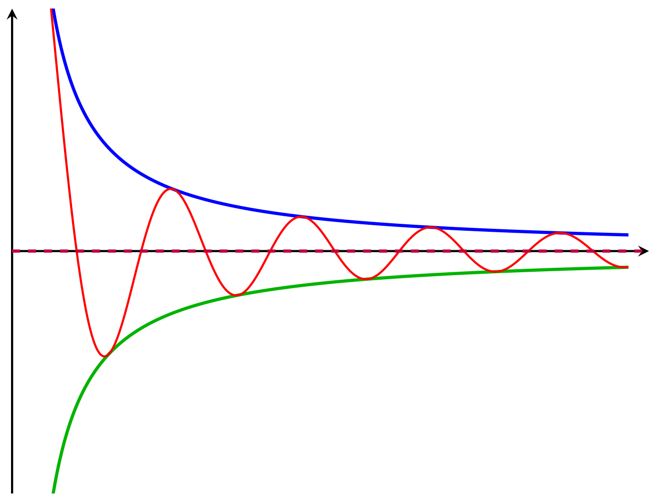
Geometric intuition: If (y_n) is "squeezed" between two sequences that converge to the same limit, it must converge to that limit too.



Exercise: Prove the Squeeze Theorem.
Hint: Use the $\varepsilon - N$ argument for $(x_n) \to \ell$ and $(z_n) \to \ell$ and then conclude $(y_n) \to \ell$.

Example: Use the Squeeze Theorem to prove that $\lim_{n\to\infty} \frac{\sin(n)}{n} = 0$.

Solution:



Activity

Problem 1: Let $(x_n) \to x$ where $x > 0$. Prove that there exists $N \in \mathbb{N}$ such that $x_n > 0$ for all $n \geq N$.
Hint: Use $\varepsilon = x/2$ in the definition of convergence.

Problem 2: Prove that if $(a_n) \to a$, then $(|a_n|) \to |a|$.
Hint: Use the reverse triangle inequality: $\big||x| - |y|\big| \leq |x - y|$.

Problem 3: Let (a_n) be a bounded sequence and assume $\lim_{n\to\infty} b_n = 0$. Prove that

$$\lim_{n\to\infty} (a_n b_n) = 0$$

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