# The Cauchy Criterion

In this handout, we tackle a deep question: *Can we tell whether a sequence converges without knowing what it converges to*? The answer is yes–and the key idea is due to a French mathematician Cauchy. A sequence converges if and only if its terms eventually cluster together, regardless of whether we know the limit. This is the **Cauchy Criterion**, and it gives us a powerful test for convergence.

**Question:** Define a sequence recursively by $a_1 = 0,\ a_2 = 1$, and

$$a_{n+1} = \frac{a_n + a_{n-1}}{2}, \quad \text{for all } n \geq 2.$$

Compute the first few terms: $a_1 = 0$, $a_2 =$ ________, $a_3 =$ ________, $a_4 =$ ________

What is the limit of this sequence?

The sequence is __________ monotone, so the Monotone Convergence Theorem doesn't directly apply. How could we prove convergence *without* knowing the limit in advance?

The Monotone Convergence Theorem is one tool – but it requires monotonicity. The Cauchy Criterion works for *any* sequence, monotone or not. The idea: instead of measuring distance to the limit, measure how close the terms are *to each other*.

## Cauchy Sequences:

**Definition 1.** A sequence $(a_n)$ is called a **Cauchy sequence** if, for every __________, there exists __________ such that whenever __________ it follows that

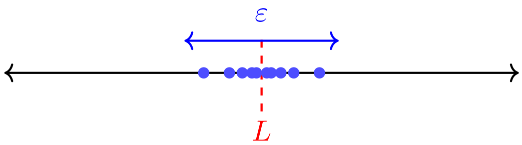
____________________

In other words, the terms of the sequence become ________________________ to each other.

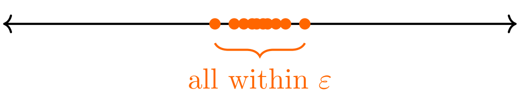
**Compare with the definition of convergence:**

| Convergence to $L$ | Cauchy |
|---|---|
| $\forall\,\varepsilon > 0,\ \exists\, N \in \mathbb{N}$ such that $n \geq N \implies \lvert a_n - L \rvert < \varepsilon$ | $\forall\,\varepsilon > 0,\ \exists\, N \in \mathbb{N}$ such that $m, n \geq N \implies \lvert a_n - a_m \rvert < \varepsilon$ |
| Requires knowing $L$ | No mention of a limit! |

**Convergent:** terms cluster near $L$




**Cauchy:** terms cluster near *each other*



**Example 1:** Show that $(a_n) = \frac{1}{n}$ is a Cauchy sequence directly from the definition.

*Scratch work:* For $m, n \geq N$, we need $|a_n - a_m| < \varepsilon$. We have:

$$|a_n - a_m| = \_\_\_\_\_\_\_\_\_\_ \leq \_\_\_\_\_\_\_\_\_\_ \leq \_\_\_\_\_\_\_\_\_\_$$

So choose $N$ such that __________.

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*Proof sketch:*

**Example 2:** Show that the sequence $a_n = (-1)^n$ is **not** a Cauchy sequence.

## Convergent $\Rightarrow$ Cauchy:

**Theorem 1.** Every convergent sequence is a Cauchy sequence.

*Proof sketch:* Suppose $(a_n) \to L$. Let $\varepsilon > 0$ be given.

Write down what it means for $(a_n) \to L$ using the $\varepsilon$-$N$ definition. (*Hint:* You may want to use $\varepsilon/2$ instead of $\varepsilon$. Why?)

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Now for $m, n \geq N$, apply the **triangle inequality** to estimate $|a_n - a_m|$.

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Combine your estimates to conclude that for $m, n \geq N$, $|a_n - a_m| < \varepsilon$.

Next, we would like to prove the converse – every Cauchy sequence is convergent. Proving the converse is a bit more challenging, mainly because, in order to prove that a sequence converges, we need to have a proposed limit of the sequence. Our strategy will be to first show that Cauchy sequences are bounded, then use the Bolzano–Weierstrass Theorem to extract a convergent subsequence whose limit becomes our candidate. We start by proving the following lemma:

**Lemma 1.** Every Cauchy sequence is bounded.

*Proof sketch:* Let $(a_n)$ be a Cauchy sequence.

**Step 1:** Choose $\varepsilon = 1$. Write down the Cauchy condition corresponding to $\varepsilon = 1$.

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Use this to show that $|a_n| < |a_N| + 1$ for all $n \geq N$.

**Step 2:** Now handle the finitely many terms $a_1, a_2, \ldots, a_{N-1}$. Define an appropriate bound $M$ that works for *all* $n \in \mathbb{N}$.

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**Compare with:** We proved a very similar result earlier in the course. Which theorem?

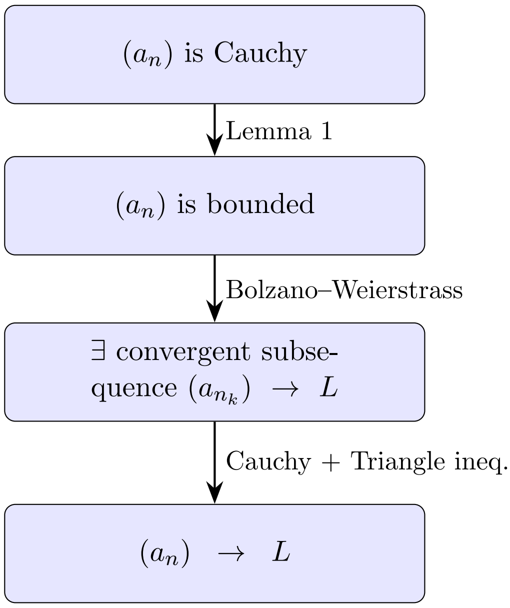
## The Cauchy Criterion:

**Theorem 2** (Cauchy Criterion)**.** A sequence converges if and only if it is a __________________ sequence.

The forward direction (⇒) is Theorem 1. The real work is the converse:

### Proof that Cauchy ⇒ Convergent:

**Strategy:** We don't know the limit, so we must *find* one. Here is the game plan:




*Proof:* Let $(a_n)$ be a Cauchy sequence.

**Step 1:** Explain why $(a_n)$ is bounded and why the Bolzano–Weierstrass Theorem gives a convergent subsequence $(a_{n_k}) \to L$.

**Step 2:** We claim $(a_n) \to L$. Let $\varepsilon > 0$ be given. Write down what the Cauchy condition gives you (with $\varepsilon/2$).

Write down what $(a_{n_k}) \to L$ gives you (with $\varepsilon/2$).

**Step 3:** Choose a specific term $a_{n_k}$ from the subsequence – pick $k$ large enough so that $k \geq K$ **and** $n_k \geq N_1$. Explain why this is possible.

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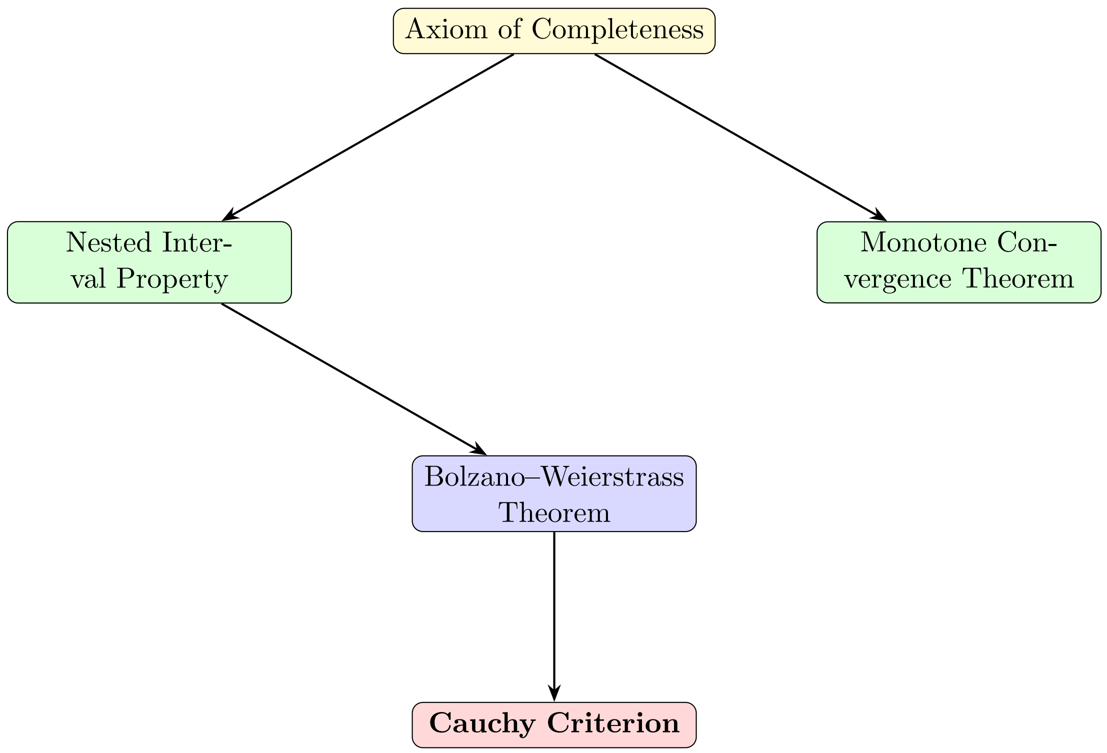
Now for any $n \geq N$, apply the triangle inequality to show $|a_n - L| < \varepsilon$.

Therefore $(a_n) \to L$. $\square$

**Reflection:** Notice how three big theorems worked together:




All of these are really statements about the ____________________ of $\mathbb{R}$.

## Activity:

**Exercise 1.** Define a sequence by $a_1 = 0,\ a_2 = 1$, and

$$a_{n+1} = \frac{a_n + a_{n-1}}{2}, \quad \text{for all } \ n \geq 2.$$

(a) Compute $a_3, a_4, a_5, a_6$.

(b) Is $(a_n)$ monotone? Explain why MCT does not directly apply.

(c) Show that $|a_{n+1} - a_n| = \frac{1}{2}|a_n - a_{n-1}|$ for all $n \geq 2$.

(d) Conclude that $|a_{n+1} - a_n| = \frac{1}{2^{n-1}}$ for all $n \geq 1$.

(e) Prove that $(a_n)$ is a Cauchy sequence.
*Hint:* For $m > n$, write $|a_m - a_n| \leq |a_{n+1} - a_n| + |a_{n+2} - a_{n+1}| + \cdots + |a_m - a_{m-1}|$ and use the finite geometric sum formula $\displaystyle\sum_{k=0}^{N} r^k = \frac{1 - r^{N+1}}{1 - r}$ for $r = \frac{1}{2}$.

(f) By the Cauchy Criterion, $(a_n)$ converges.