The Monotone Convergence Theorem and Infinite Series

In this handout, we explore one of the most powerful convergence tools in analysis: sequences that are both bounded and monotone always converge! We then use this to understand when infinite series converge.

Question: We know that convergent sequences are bounded (Theorem 1 of a previous handout). But is the converse true? Does every bounded sequence converge?

So boundedness alone is **not enough** for convergence. But what if we add another condition?

Monotone Sequences:

Definition 1. A sequence (a_n) is called:

- **increasing** if a_n ________ a_{n+1} for all $n \in \mathbb{N}$.
- **decreasing** if a_n ________ a_{n+1} for all $n \in \mathbb{N}$.

A sequence is **monotone** if it is either ____________________ or ____________________.

Examples: Determine if each sequence is increasing, decreasing, or neither.

(a) $(a_n) = \left(\frac{n}{n+1}\right) = \left(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\right)$

Answer: ____________________

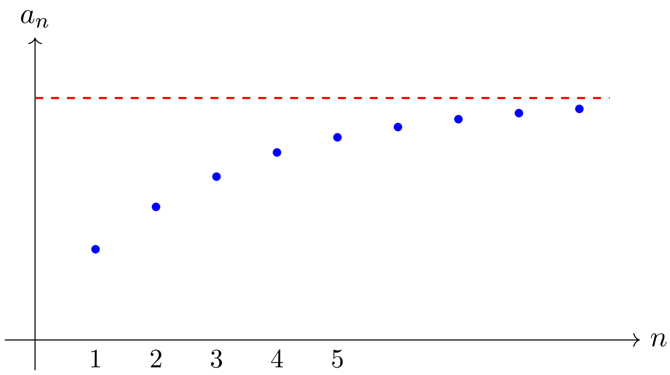
(b) $(b_n) = \left(\frac{1}{n}\right) = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right)$

Answer: ____________________

(c) $(c_n) = ((-1)^n)$

Answer: ____________________

Theorem 1 (Monotone Convergence Theorem)**.** If a sequence is ____________________ and _______________, then it converges.



Proof sketch: Let (a_n) be increasing and bounded.
Step 1: Find a candidate for the limit.

Since (a_n) is bounded, *explain* why the set $\{a_n : n \in \mathbb{N}\}$ is *bounded above.*

so by ______________________________, the set $\{a_n : n \in \mathbb{N}\}$ has a ______________________.
Let:

$$s = \sup\{a_n : n \in \mathbb{N}\}$$

We claim that $\lim_{n\to\infty} a_n =$ __________.
Step 2: Prove $a_n \to s$ using the definition of convergence.

Write the $\varepsilon - N$ definition of $a_n \to s$.

Since s is the ______________________________, $s - \varepsilon$ is __________ an upper bound.
Therefore, there exists a_N such that ____________________ $< a_N$.

For $n \geq N$, use the fact that the sequence (a_n) is increasing, and show that $|a_n - s| < \varepsilon$.

Therefore $|a_n - s| < \varepsilon$. $\square$

Example: Let $x_1 = 2$ and $x_{n+1} = \frac{1}{2}(x_n + 1)$ for $n \geq 1$. Show that (x_n) converges. Then find $\lim_{n\to\infty} x_n$.

First, let's compute a few terms:

$x_1 = 2$, $x_2 =$ __________, $x_3 =$ __________

The sequence appears to be ____________________.

Proving (x_n) is decreasing: We want to show $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$.

Proving (x_n) is bounded below:

By the Monotone Convergence Theorem, (x_n) converges to some limit L. Taking the limit on both sides of $x_{n+1} = \frac{1}{2}(x_n + 1)$:

Introduction to Infinite Series:

We want to make sense of expressions like:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

or

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$$

Do these "infinite sums" have finite values?

Definition 2. Let (b_n) be a sequence. An **infinite series** is a formal expression of the form

$$\sum_{n=1}^{\infty} b_n = \underline{\hspace{5cm}}$$

We define the **sequence of partial sums** (s_m) by

$$s_m = \underline{\hspace{4cm}} = \underline{\hspace{5cm}}$$

We say that the series $\sum_{n=1}^{\infty} b_n$ **converges to** B if the sequence (s_m) ____________________ to B. In this case, we write:

____

So, Understanding series is really about understanding __________________________.

Example: Consider the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$

The mth partial sum is: $s_m =$ ____________________________________

Question 1: Is the sequence (s_m) monotone? _______________

Since all terms $\frac{1}{n^2} > 0$, we have $s_1 < s_2 < s_3 < \cdots$, so (s_m) is ____________________.

Question 2: Is (s_m) bounded above?

Let's estimate:

$$s_m = 1 + \frac{1}{2 \cdot 2} + \frac{1}{3 \cdot 3} + \frac{1}{4 \cdot 4} + \cdots + \frac{1}{m \cdot m}$$

$$< \underline{\hspace{8cm}}$$

$$= \underline{\hspace{8cm}}$$

Using the telescoping formula: $\frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$

Therefore $s_m <$ __________ for all m.

Since the sequence (s_m) is *bounded above* and *monotone*, by Monotone Convergence theorem, (s_m) is convergent and so $\sum_{n=1}^{\infty} \frac{1}{n^2}$ ____________________________.

(The actual sum turns out to be $\frac{\pi^2}{6}$, discovered by Euler!)

The Harmonic Series:

Example: Does the infinite series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ converge?

The partial sums $s_m = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}$ are ____________________.

Are they bounded? Let's investigate...

Consider:

$$s_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + (\_\_\_\_ + \_\_\_\_) = \_\_\_\_$$

$$s_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + (\_\_\_\_\_\_\_\_\_\_\_\_) = \_\_\_\_\_\_\_\_$$

$$s_{16} > \_\_\_\_\_\_\_\_$$

In general, $s_{2^k} > 1 + \frac{k}{2}$, which is ____________________ as $k \to \infty$.

Conclusion: The harmonic series ____________________.

The Cauchy Condensation Test

Theorem 2 (Cauchy Condensation Test)**.** Suppose (b_n) is ____________________ and satisfies $b_n \geq 0$ for all $n \in \mathbb{N}$. Then the series $\sum_{n=1}^{\infty} b_n$ converges if and only if the series

$$\sum_{n=0}^{\infty} 2^n b_{2^n} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

converges.

This is a powerful tool for testing series with decreasing terms!

Example: Use the Cauchy Condensation Test on the harmonic series.
For $b_n = \frac{1}{n}$:

$$\sum_{n=0}^{\infty} 2^n b_{2^n} = \sum_{n=0}^{\infty} 2^n \cdot \frac{1}{2^n} = \sum_{n=0}^{\infty} \_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

This series ____________________, so the harmonic series ____________________.

The p-Series Test:

Corollary 1. The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if ________________.

This can be proved using the Cauchy Condensation Test and properties of geometric series!

Your turn: Determine whether each series converges or diverges.

(a) $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

(b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

(c) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

Activity

Exercise: Let $x_1 = 1$ and $x_{n+1} = \sqrt{2 + x_n}$ for $n \geq 1$.
(a) Compute x_2, x_3, x_4 and conjecture whether the sequence is increasing or decreasing.

(b) Prove the sequence is increasing. [Hint: Use induction to show $x_n < 2$ for all n, then show $x_{n+1} > x_n$.]

(c) Prove the sequence is bounded above by 2.

(d) Find $\lim x_n$.