Subsequences and the Bolzano-Weierstrass Theorem

In this handout, we explore a fundamental concept: every bounded sequence has a convergent subsequence! This powerful result, called the Bolzano-Weierstrass Theorem, is one of the cornerstones of analysis.

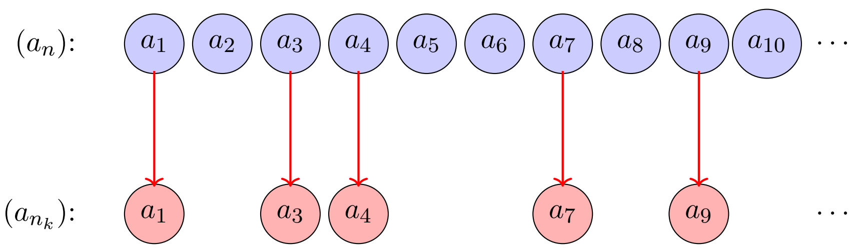
Question: Consider the sequence $(a_n) = ((-1)^n) = -1, 1, -1, 1, -1, 1, \ldots$
Does this sequence converge?

Even though the full sequence doesn't converge, can you find a "part" of this sequence that does converge?

This motivates the idea of **subsequences**: extracting terms from a sequence while preserving their order.

Subsequence:

Definition 1. Let (a_n) be a sequence. A **subsequence** is a sequence of the form (a_{n_k}) where $n_1 < n_2 < n_3 < \cdots$ is a ______________________________ sequence of natural numbers. In other words, we select terms from (a_n) in the same order they appear, but we can skip terms.



Example: Consider the sequence $(a_n) = \left(\frac{1}{n}\right) = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

(a) Is $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots\right)$ a subsequence? If yes, what is n_k?

(b) Is $\left(1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots\right)$ a subsequence? If yes, what is n_k?

(c) Is $\left(\frac{1}{2}, 1, \frac{1}{3}, \frac{1}{4}, \ldots\right)$ a subsequence? Why or why not?

(d) Is $\left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \ldots\right)$ a subsequence?

Important Observation: If $n_1 < n_2 < n_3 < \cdots$, then what can we say about the relationship between k and n_k?
Hint: Consider the first few terms. We have $n_1 \geq$ _____, $n_2 \geq$ _____, $n_3 \geq$ _____, etc.

In general: $n_k \geq$ __________ for all $k \in \mathbb{N}$ – we can formally prove this using Mathematical induction!
This means as $k \to \infty$, we also have $n_k \to$ __________.

Convergence of Subsequences:

Theorem 1. If $(a_n) \to L$, then every subsequence (a_{n_k}) also converges to L.

Proof sketch: Let $(a_n) \to L$ and let (a_{n_k}) be a subsequence. Let $\varepsilon > 0$ be given.

Write down what it means for $(a_n) \to L$ using the ε-N definition.

To show the subsequence $(a_{n_k}) \to L$, we need to show that there exists $K \in \mathbb{N}$ such that whenever ________, it follows that

____

Choose $K = N$. For all $k \geq K = N$, explain why $n_k \geq N$.

Now complete the proof by showing $|a_{n_k} - L| < \varepsilon$.

□

Important: If a sequence has a subsequence that converges to L_1 and another subsequence that converges to L_2 where $L_1 \neq L_2$, what can we conclude about the original sequence?

Example: Show that $((-1)^n)$ diverges using this idea.

The Bolzano-Weierstrass Theorem:

Now we arrive at one of the most important theorems in analysis!

Theorem 2. Every ____________________ sequence has a _______________________ subsequence.

This is remarkable! Even if a sequence doesn't converge, as long as it's bounded, we can *always* find a convergent piece of it.

Before the proof: Let's build intuition.
Consider the bounded sequence: $(a_n) = 1, 0, -1, 0, 1, 0, -1, 0, \ldots$

(a) Does the sequence (a_n) converge?

(b) Can you find a convergent subsequence? What does it converge to?

(c) Can you find another convergent subsequence that converges to a different limit?

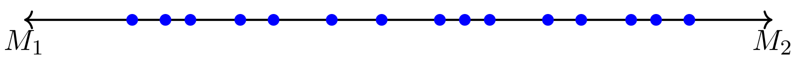
Proof Idea: The Bisection Method

The proof uses a clever bisection argument. Let's build it step by step!

Setup: Suppose (a_n) is a bounded sequence. Then there exist M_1, M_2 such that

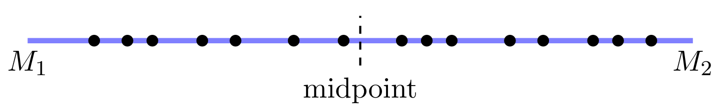
$$M_1 \leq a_n \leq M_2 \quad \text{for all } n \in \mathbb{N}.$$

Let's visualize some terms of this sequence on the number line:



Since the sequence is **bounded**, all infinitely many terms are trapped in this interval!

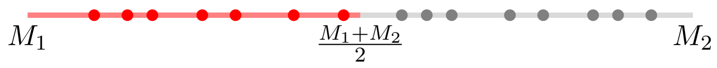
Proof sketch: **Step 1:** Start with $I_1 = [M_1, M_2]$ containing all terms of (a_n).



Now divide I_1 into two equal halves: $\left[M_1, \frac{M_1+M_2}{2}\right]$ and $\left[\frac{M_1+M_2}{2}, M_2\right]$.

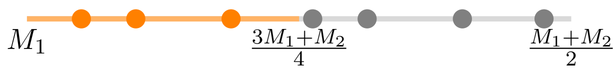
At least one of these two halves must contain infinitely many terms of (a_n). Why? *Hint:* Think about contradiction!

Choose one half that contains infinitely many terms and call it $I_2 = [a_2, b_2]$.



Pick any term from this interval, say $a_{n_1} \in I_2$. Note that $|I_2| =$ __________.

Step 2: Repeat the process!
Now divide I_2 into two equal halves and choose a half I_3 that contains infinitely many terms. Call it $I_3 = [a_3, b_3]$.



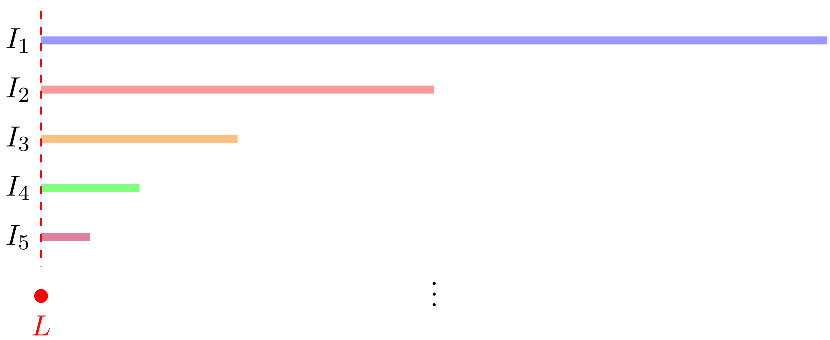
Pick a term $a_{n_2} \in I_3$ with $n_2 > n_1$.

Wait, why can we find $n_2 > n_1$?

Note that $|I_3| =$ ____________________ $=$ ______________.

Step 3: Continue indefinitely!
At each stage k, we have:

- An interval $I_k =$ __________ with length __________________.
- I_k contains ______________ many terms of (a_n)
- We've chosen a term $a_{n_{k-1}} \in I_k$ with $n_1 < n_2 < n_3 < \cdots < n_{k-1}$



What happens to the length of I_k as $k \to \infty$?

$$\text{Length of } I_k = |I_k| = \frac{M_2 - M_1}{2^{k-1}} \to \underline{\hspace{3cm}} \text{ as } k \to \infty$$

By the **Nested Interval Property**, the intersection $\bigcap_{k=1}^{\infty} I_k$ contains a ___________ point L.

Step 4: The subsequence (a_{n_k}) converges to L!
Recall we chose $a_{n_k} \in I_k$ for each k. Let's prove why $(a_{n_k}) \to L$.

Let $\varepsilon > 0$ be given. We need to show that there exists $K \in \mathbb{N}$ such that whenever $k \geq K$, it follows that

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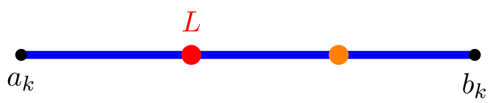
Since $a_{n_k} \in I_k$ and $L \in I_k$, we have:

$$|a_{n_k} - L| \leq \underline{\hspace{6cm}} = \underline{\hspace{4cm}}$$



Now, choose K large enough so that $\dfrac{M_2 - M_1}{2^{K-1}} < \varepsilon$. We can do this because:

Then for all $k \geq K$:

$$|a_{n_k} - L| \leq \frac{M_2 - M_1}{2^{k-1}} \leq \underline{\hspace{5cm}} < \underline{\hspace{3cm}}$$

Therefore, $(a_{n_k}) \to L$!

□