Properties of Infinite Series

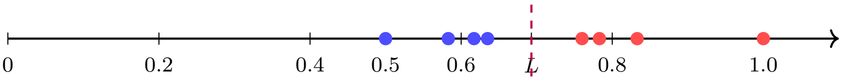
So far, we have developed powerful tools for sequences. Now we turn to **infinite series** again – sums of infinitely many real numbers. This handout develops the core toolkit – algebraic rules, convergence tests, and the distinction between absolute and conditional convergence – which leads to one of the most striking theorems in all of real analysis.

Question: Consider the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

Compute the first several partial sums:

n	s_n
1	1
2	$1 - \frac{1}{2} = 0.500$
3	$1 - \frac{1}{2} + \frac{1}{3} \approx 0.833$
4	$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \approx 0.583$
5	≈ 0.783
6	≈ 0.617
7	≈ 0.760
8	≈ 0.635



Now consider the rearrangement:

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots$$

Does this rearrangement converge to the same L? To a different value? To anything at all?

Recall Infinite Series:

Given $\sum_{k=1}^{\infty} a_k$, we always track *two* sequences:

Sequence of terms	**Sequence of partial sums**
$(a_1, a_2, a_3, \ldots)$	$s_n = a_1 + a_2 + \cdots + a_n$
The “ingredients”	The “running totals”

Definition: We say $\sum_{k=1}^{\infty} a_k = A$ if and only if $\lim_{n\to\infty} s_n = \underline{\hspace{3cm}}$.

Example 1. Does the series converge or diverge?

1. $\sum_{k=1}^{\infty} \frac{1}{k^2}$
2. $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$

Algebraic Rules for Series:

Theorem 1. If $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$, then:

(i) $\sum_{k=1}^{\infty} c\, a_k = cA$ for all $c \in \mathbb{R}$, and

(ii) $\sum_{k=1}^{\infty} (a_k + b_k) = A + B$.

Proof of (ii). Let $s_n = a_1 + a_2 + \cdots + a_n$ and $t_n = b_1 + b_2 + \cdots + b_n$ be the partial sums of $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ respectively.

Let u_n denote the n-th partial sum of $\sum_{k=1}^{\infty}(a_k + b_k)$. Then:

$$u_n = \sum_{k=1}^{n} (a_k + b_k) = \underline{\hspace{6cm}}$$

Since $\sum_{k=1}^{\infty} a_k = A$, we know $\lim_{n\to\infty} s_n = \underline{\hspace{2cm}}$.

Since $\sum_{k=1}^{\infty} b_k = B$, we know $\lim_{n\to\infty} t_n = \underline{\hspace{2cm}}$.

Therefore:

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} (s_n + t_n) = \underline{\hspace{3cm}} + \underline{\hspace{3cm}} = \underline{\hspace{1.5cm}} + \underline{\hspace{1.5cm}}.$$

where the last step follows from $\underline{\hspace{5cm}}$ of limits. ☐

The Cauchy Criterion for Series:

Because convergence of a series *means* convergence of its partial sums, the Cauchy Criterion for sequences immediately yields a criterion for series.

Theorem 2 (Cauchy Criterion for Series)**.** The series $\sum_{k=1}^{\infty} a_k$ converges if and only if, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that whenever $n > m \geq N$, it follows that

$$\left|a_{m+1} + a_{m+2} + \cdots + a_n\right| < \varepsilon.$$

Proof Sketch: Recall that $\sum_{k=1}^{\infty} a_k$ converges if and only if the sequence of partial sums (s_n) is ________________.

Write the definition of the sequence (s_n) being Cauchy.

Now expand $|s_n - s_m|$. By definition of partial sums, for $n > m$:

$s_n - s_m =$ ________________ $=$ ________________________________ $=$ ___________________________.

Therefore:

$|s_n - s_m| =$ ___________________________.

Putting this together:

($\Rightarrow$) Suppose $\sum_{i=1}^{\infty} a_k$ converges. Then (s_n) converges, hence (s_n) is ___________. Therefore for every $\varepsilon > 0$, there exists N such that $n > m \geq N$ implies $|s_n - s_m| < \varepsilon$, which means:

____

($\Leftarrow$) Suppose for every $\varepsilon > 0$ there exists N such that $n > m \geq N$ implies $|a_{m+1} + \cdots + a_n| < \varepsilon$. Then $|s_n - s_m| < \varepsilon$, so (s_n) is ___________. By the Cauchy Criterion for sequences, (s_n) ________________, and therefore $\sum_{k=1}^{\infty} a_k$ converges. □

The Divergence Test:

Theorem 3. If $\sum_{k=1}^{\infty} a_k$ converges, then $(a_k) \to 0$.

Proof Sketch:

Suppose $\sum_{k=1}^{\infty} a_k$ converges. Then the sequence of partial sums (s_n) converges to some limit A. This means:

$\lim_{n\to\infty} s_n =$ ___________ and also $\lim_{n\to\infty} s_{n-1} =$ ___________.

Now write a_n in terms of partial sums. Note that:

$$a_n = s_n - \underline{\qquad\qquad}$$

Therefore:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1}) = \underline{\qquad\qquad} - \underline{\qquad\qquad} = \underline{\qquad\qquad}.$$

where we used $\underline{\qquad\qquad\qquad\qquad}$ of limits. □

Important Remark. The converse is **false**. Give an example of a series $\sum_{n=1}^{\infty} a_k$ where $a_k \to 0$ but the series *diverges*:

Example 2: Use the Term Test to determine whether each series *could* converge (C) or definitely diverges (D). Explain.

(a) $\sum_{n=1}^{\infty} \frac{n}{n+1}$ $\underline{\qquad}$

(b) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ $\underline{\qquad}$

(c) $\sum_{n=1}^{\infty} \sin(n)$ $\underline{\qquad}$

Comparison Test:

Theorem 4 (Comparison Test)**.** Assume (a_k) and (b_k) satisfy $0 \le a_k \le b_k$ for all $k \in \mathbb{N}$.

(i) If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.

(ii) If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.

Proof Sketch.

Let $s_n = \sum_{k=1}^{n} a_k$ and $t_n = \sum_{k=1}^{n} b_k$ be the partial sums of $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ respectively.

Since $0 \le a_k \le b_k$ for all k, what can you say about the relationship between s_n and t_n?

$$s_n \underline{\qquad} t_n \quad \text{for all } n \in \mathbb{N}.$$

Explain why both sequences (s_n) and (t_n) are *monotone* and *bounded.*

Proof of (i). Since, the sequence (s_n) is monotone increasing and ________________. By the Monotone Convergence Theorem, (s_n) converges, and therefore $\sum_{k=1}^{\infty} a_k$ ___________. □

Proof of (ii). Notice that (ii) is simply the ________________ of (i).

So (ii) requires no additional work — it follows immediately from (i). □

Geometric Series: The series $\sum_{k=0}^{\infty} ar^k$ converges if and only if ___________, with sum ______.

Example 3: Does $\sum_{n=1}^{\infty} \frac{1}{2^n + n}$ converge?

Absolute vs. Conditional Convergence:

This is one of the most important distinctions in all of series theory.

Theorem 5 (Absolute Convergence Test)**.** If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges as well.

Proof Sketch. Let $\sum_{n=1}^{\infty} |a_n|$ be a convergent series. Then the partial sum sequence

$$(s_m) = \underline{\hspace{6cm}}$$

converges and hence is Cauchy.

Write what it means to say that the sequence (s_m) is Cauchy.

Use the *triangle inequality* to show that the partial sum sequence for the series $\sum_{n=1}^{\infty} a_n$ is Cauchy, and conclude that $\sum_{n=1}^{\infty} a_n$ converges.

Therefore $\sum_{n=1}^{\infty} a_n$ converges. ☐

The Alternating Series Test:

Many naturally occurring series have alternating signs. The following test handles this case elegantly.

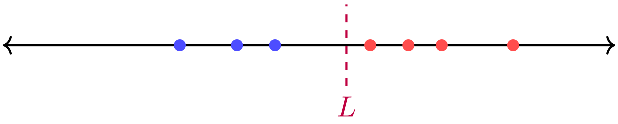
Theorem 6 (Alternating Series Test)**.** Let (a_n) be a sequence satisfying:

(i) $a_1 \geq a_2 \geq a_3 \geq \cdots$, and

(ii) $(a_n) \to 0$.

Then $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

Geometric intuition: The partial sums “zigzag” toward the limit.



The even partial sums form an ____________ sequence; the odd partial sums form a ____________ sequence. They squeeze toward the same limit.

Proof Sketch. Let s_n denote the n-th partial sum of $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$, that is:

$$s_n = \sum_{k=1}^{n} (-1)^{k+1} a_k = a_1 - a_2 + a_3 - \cdots + (-1)^{n+1} a_n.$$

We will prove this theorem in steps.

Observation: The hypotheses of the theorem state that (a_n) is decreasing and $a_n \to 0$. Non-negativity $a_n \geq 0$ is *not* explicitly assumed, but explain below why it follows from the two given conditions.

Step 1. Show that the sequence (s_{2n}) is *increasing.*

Step 2. Show that the sequence (s_{2n}) is *bounded above.*

Step 3. By the ____________________________________ Theorem, $(s_{2n}) \to L$ for some $L \in \mathbb{R}$.

Show that (s_{2n+1}) also converges to L.

Conclude that $(s_n) \to L$ as $n \to \infty$.

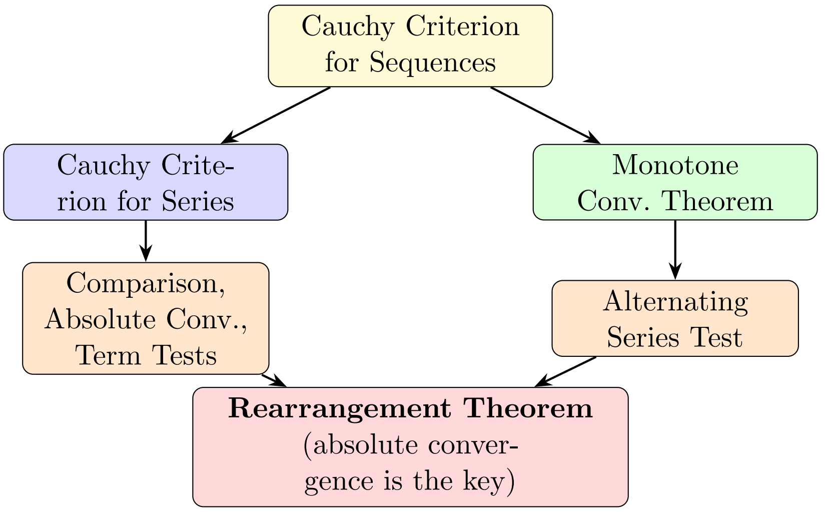
Therefore $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges. □

Rearrangements:

Now we can return to the question at the start of this handout.

Theorem 7 (Rearrangement Theorem)**.** If a series converges absolutely, then any rearrangement of the series converges to the same limit.

The Big Picture – How all the tools connect:



Activity

Exercise 1. For each series, determine convergence or divergence. State which test you use and verify its hypotheses. For the series in part (b), does it converge absolutely or conditionally?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$

Exercise 2. Suppose $\displaystyle\sum_{k=1}^{\infty} a_k = A$ and $\displaystyle\sum_{k=1}^{\infty} b_k$ diverges.

(a) What can you conclude about $\displaystyle\sum_{k=1}^{\infty} (a_k + b_k)$? Prove your claim.

(b) Is it possible for $\displaystyle\sum_{k=1}^{\infty} a_k b_k$ to converge? Give an example or prove impossibility.