Open Sets, Closed Sets, and Limit Points

In Chapter 2, we learned that sequences are a powerful tool for studying convergence in $\mathbb{R}$. But now we want to ask the question: rather than asking *"does this sequence converge"?*, we ask *"what kind of set is this"?*

Question: Which subsets $A \subseteq \mathbb{R}$ have the property that every convergent sequence (a_n) with $a_n \in A$ satisfies $\lim_{n\to\infty} a_n \in A$?

Let's think about some examples:

- Does $(0, 1)$ have this property? ____________________ Give a sequence that supports your answer.
- Does $[0, 1]$ have this property? ____________________

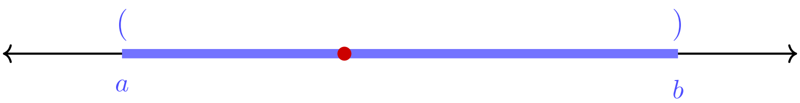
These concepts form the foundational vocabulary of topology on $\mathbb{R}$, and will reappear throughout the rest of the course – in the study of continuity, differentiation, and integration.

Open Sets:

Definition 1. A set $O \subseteq \mathbb{R}$ is called **open** if for every point $a \in O$, there exists an $\varepsilon > 0$ such that the ε-neighborhood

$$V_\varepsilon(a) = \underline{\hspace{6cm}}.$$

Intuitively, a set is open if every point has "elbow room" – no matter where you stand inside O, you can wiggle a little in either direction and stay inside O.



Your turn: For each set, determine whether it is open using the definition. If **open**, exhibit an explicit $\varepsilon > 0$ (in terms of x) such that $V_\varepsilon(x) \subseteq O$ and verify the inclusion.

(a) $O = (1, 5)$ Open? ____________.

(b) $O = (1,5) \cup (7,9)$ Open? ________.

(c) $O = \{3\}$ Open? ________.

(d) $O = \emptyset$ Open? ________.

(e) $O = [1,5)$ Open? ________.

Theorem 1.

(i) The union of *any* collection of open sets is open: if $\{O_\lambda\}_{\lambda\in\Lambda}$ is any collection of open sets, then $\bigcup_{\lambda\in\Lambda} O_\lambda$ is open.

(ii) The intersection of *finitely many* open sets is open: if $O_1, \ldots, O_n$ are open, then $\bigcap_{k=1}^{n} O_k$ is open.

(iii) The sets $\mathbb{R}$ and $\emptyset$ are open.

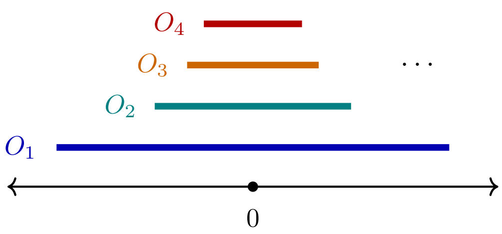
Proof of (i): Let $\{O_\lambda\}_{\lambda\in\Lambda}$ be any collection of open sets and let $x \in \bigcup_{\lambda\in\Lambda} O_\lambda$.

Prove that $\bigcup_{\lambda\in\Lambda} O_\lambda$ is open.

Proof of (ii): Let $O_1, \ldots, O_n$ be open and let $x \in \bigcap_{k=1}^{n} O_k$.

Show that $\bigcap_{k=1}^{n} O_k$ is open.

Note: Part (ii) of Theorem 1 *may* fail for infinite intersections. Consider $O_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$ for $n = 1, 2, 3, \ldots$



What is $\bigcap_{n=1}^{\infty} O_n$? ______________________

Is this set open? _________________

Limit Points:

Before defining closed sets, we need a key new concept that bridges Chapter 2 and Chapter 3.

Definition 2. A point $x \in \mathbb{R}$ is a **limit point** (or **accumulation point**) of a set $A \subseteq \mathbb{R}$ if *every* ε-neighborhood $V_\varepsilon(x)$ contains a point of A *different from x itself*:

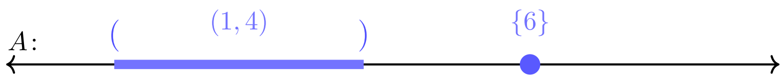
____.

....

A point $a \in A$ that is not a limit point of A is called an ________________________ of A.

Note:

1. Think of a limit point as a "popular" point – no matter how small a neighborhood you draw around x, there is always at least one point of A nearby.
2. Limit points do *not* need to be in A! – can be inside *or* outside the set. The picture below shows all three cases for the set $A = (1, 4) \cup \{6\}$:



Your turn: For each set A, find all limit points and call the collection $L(A)$.

(a) $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$ $L(A) =$ ________________

(b) $A = (0, 1)$ $L(A) =$ ________________

(c) $A = \mathbb{Z}$ $L(A) =$ ________________

The following theorem connects limit points back to Chapter 2 and is extremely useful in practice:

Theorem 2. A point x is a limit point of a set A if and only if there exists a sequence (a_n) with each $a_n \in A$ and $a_n \neq x$, such that $(a_n) \to x$.

Proof sketch:

($\Leftarrow$) Suppose $(a_n) \to x$ with $a_n \in A$, $a_n \neq x$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$.

Use the fact that $(a_n) \to x$ to show that $V_\varepsilon(x)$ contains a point of A different from x.

Therefore, x is a limit point of A. $\square$

($\Rightarrow$) Suppose x is a limit point of A.

For each $n \in \mathbb{N}$, since x is a limit point of A, apply the definition with $\varepsilon = \frac{1}{n}$:

Show that $(a_n) \to x$:

Closed Sets:

Definition 3. A set $F \subseteq \mathbb{R}$ is called **closed** if it contains all of its limit points:

$$x \text{ is a limit point of } F \implies \underline{\qquad\qquad}.$$

Your turn: Identify the limit points and check whether they are all inside F.

(a) $F = [1, 5]$ Closed? ____________

(b) $F = (1, 5]$ Closed? ____________

(c) $F = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$ Closed? ____________

(d) $F = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \{0\}$ Closed? ____________

(e) $F = \mathbb{Z}$ Closed? ____________

The following theorem answers the motivating question from the very beginning of this handout!

Theorem 3. A set $F \subseteq \mathbb{R}$ is closed if and only if: whenever (a_n) is a convergent sequence with each $a_n \in F$, the limit satisfies $\lim_{n\to\infty} a_n \in F$.

$$(a_n) \subseteq F \text{ and } (a_n) \to L \implies L \in F.$$

Proof sketch:

($\Rightarrow$) Suppose F is closed and $(a_n) \subseteq F$ with $(a_n) \to L$.

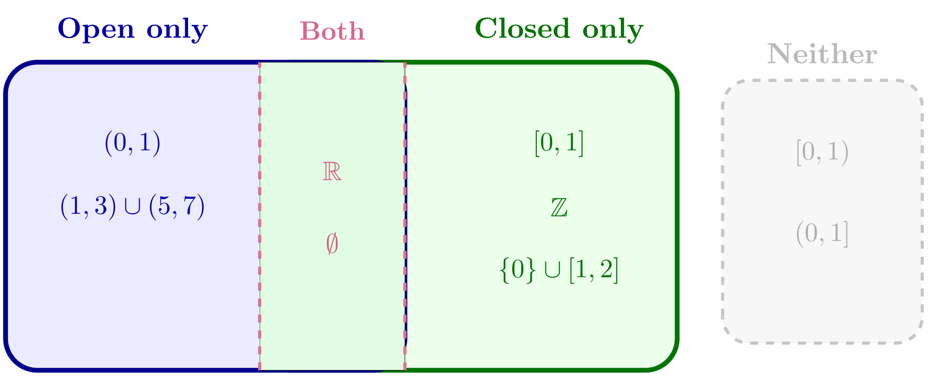
Case 1: $a_n = L$ for some $n \in \mathbb{N}$. Why does this immediately give $L \in F$?

Case 2: $a_n \neq L$ for all $n \in \mathbb{N}$. Use Theorem 2 to conclude that L is a *limit point* of F. Then conclude $L \in F$:

($\Leftarrow$) Suppose F is *not* closed. What does this mean? That is, what is the negation of "F contains all its limit points"? Can you get a *contradiction*?

Open vs. Closed:

Students often assume that "open" and "closed" are opposites. They are **not**! A set can be both, neither, or exactly one.



The algebraic link between open and closed is through complements:

Theorem 4. A set $F \subseteq \mathbb{R}$ is closed if and only if its complement $F^c = \mathbb{R} \setminus F$ is open.

Proof sketch: ($\Rightarrow$) Suppose F is closed and let $x \in F^c$, so $x \notin F$.

Since F is closed and $x \notin F$, the point x is **not** a limit point of F. Write out explicitly what this means:

Use this to find an $\varepsilon > 0$ such that $V_\varepsilon(x) \subseteq F^c$, concluding that F^c is open:

($\Leftarrow$) Suppose F^c is open. We want to show F is closed, i.e., every limit point of F belongs to F.

Let x be a limit point of F. Suppose for contradiction that $x \in F^c$. Use the openness of F^c to find a neighborhood of x that misses F entirely, and explain why this contradicts x being a limit point of F:

Closed sets enjoy analogous properties to open sets, but with unions and intersections swapped.

Theorem 5.

(i) The sets $\mathbb{R}$ and $\emptyset$ are closed.

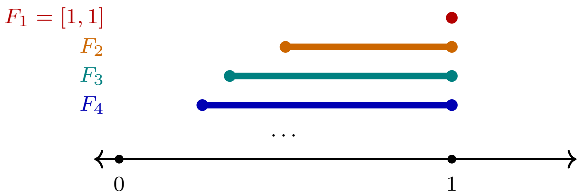
(ii) The intersection of *any* collection of closed sets is closed.

(iii) The union of *finitely many* closed sets is closed.

Proof of (ii). Let $\{F_\lambda\}_{\lambda \in \Lambda}$ be any collection of closed sets.

Use Theorem 4 and De Morgan's law to show that $\bigcap_{\lambda \in \Lambda} F_\lambda$ is closed.

Note: Part (iii) of Theorem 5 *may* fail for infinite unions. Consider $F_n = \left[\frac{1}{n}, 1\right]$ for $n = 1, 2, 3, \ldots$



What is $\bigcup_{n=1}^{\infty} F_n$? ____________________ Is this set closed? __________

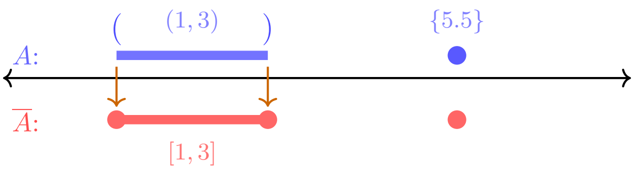
Closure of a Set:

Definition 4. Let $A \subseteq \mathbb{R}$. The **closure** of A, denoted $\overline{A}$, is defined as

$$\overline{A} = A \cup L(A),$$

where $L(A)$ is the set of all limit points of A.

Intuitively, $\overline{A}$ is the "smallest closed set containing A": we add in exactly the limit points that A is missing.



Compute the closure.

(a) $A = (0, 1)$ $\overline{A} =$ ____________________

(b) $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ $\overline{A} =$ ____________________

(c) $A = \mathbb{Z}$ $\overline{A} =$ ____________________