

Compact Sets

Throughout our study of analysis, we have worked with sequences in *infinite* sets such as $(0, 1)$, \mathbb{R} etc. A difficulty in that is all the infinite processes—limits, series, integrals—can behave badly. The concept of a **compact set** is designed precisely to impose enough structure so that infinite processes can be controlled as if they were finite.

Question: Suppose (a_n) is a sequence that *does not converge*. Can we at least guarantee some “convergent part” of it?

Recall the **Bolzano–Weierstrass Theorem**: every bounded sequence has a convergent subsequence.

- Does the sequence $a_n = (-1)^n$ converge? _____ Does it have a convergent subsequence? _____
- Does the sequence $a_n = n$ (in \mathbb{R}) have a convergent subsequence? _____ Why not?
- If a sequence takes values in $[0, 1]$, is it automatically bounded? _____ Must its subsequential limits also lie in $[0, 1]$? _____

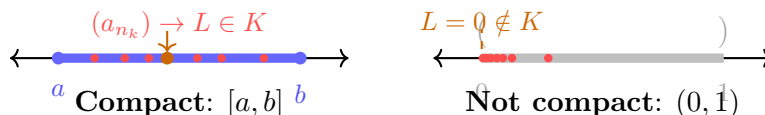
The key feature of $[0, 1]$ is: it is *bounded* (so B-W applies) *and closed* (so limits stay inside). This is exactly what compactness captures!

Definition of Compactness:

Definition 1. A set $K \subseteq \mathbb{R}$ is called **compact** if every sequence (a_n) contained in K has a subsequence (a_{n_k}) that converges to a limit that is also in K :

$$(a_n) \subseteq K \implies \exists (a_{n_k}) \text{ with } a_{n_k} \rightarrow L \in K.$$

Compactness is a “self-contained” property: not only do convergent subsequences *exist*, but their limits remain *trapped* inside K .



Your turn: For each set K , determine whether it is compact. If K is not compact, produce a sequence $(x_n) \subseteq K$ such that *every* convergent subsequence of (x_n) converges to a limit not contained in K . If K is compact, explain why every sequence in K has a subsequence converging to a limit in K .

(a) $K = [0, 1]$ Compact? _____

(b) $K = (0, 1)$

Compact? _____

(c) $K = [1, \infty)$

Compact? _____

(Hint: Can you find an unbounded sequence in K ?)(d) $K = \{0\} \cup \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$

Compact? _____

The Characterization Theorem:

The examples above reveal a pattern: compact sets in \mathbb{R} appear to be exactly the sets that are both *closed and bounded*. Let us record this precisely.

Definition 2. A set $A \subseteq \mathbb{R}$ is **bounded** if there exists $M > 0$ such that

Theorem 1. A set $K \subseteq \mathbb{R}$ is compact if and only if it is closed *and* bounded.

The proof has two parts. We show compactness implies closed and bounded, and then we show that closed and bounded implies compactness.

Proof sketch (\Rightarrow): We will prove this implication in two steps.

Step 1: Suppose, for contradiction, that K is compact but *not* bounded.

Show that for each $n \in \mathbb{N}$ there exists $x_n \in K$ with $|x_n| > n$.

Explain why the sequence (x_n) cannot have *any* convergent subsequence:

Why does this contradict compactness of K ?

Step 2: Now suppose K is compact and $(x_n) \subseteq K$ with $x_n \rightarrow L$.

Use compactness to prove that (x_n) has a convergent subsequence (x_{n_k}) with $(x_{n_k}) \rightarrow L$.

Conclude that $L \in K$. Why does this show K is closed?

Proof sketch (\Leftarrow): Let K be closed and bounded, and let $(a_n) \subseteq K$.

Prove that (a_n) has a convergent subsequence (a_{n_k}) converging to L .

Argue why $L \in K$. Does it imply K is compact?

Your turn: Decide whether each set is compact by checking if it is closed and bounded.

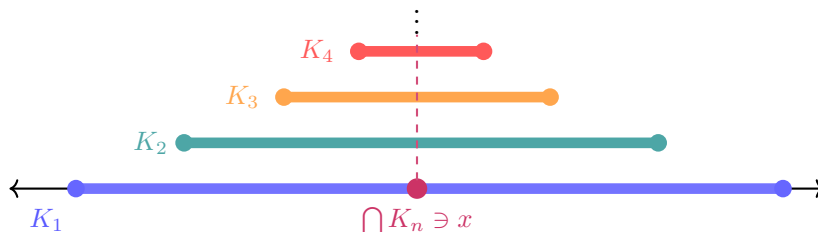
Set	Bounded?	Closed?	Compact?
$[2, 7]$			
$[2, 7)$			
\mathbb{Z}			
$\left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$			
$[0, 1] \cup [3, 5]$			

Nested Compact Sets:

Recall the Nested Interval Property from Chapter 1: a nested sequence of closed intervals $[a_1, b_1] \supseteq [a_2, b_2] \supseteq \dots$ always has nonempty intersection. Compactness allows us to generalize this property.

Theorem 2. If $K_1 \supseteq K_2 \supseteq K_3 \supseteq \dots$ is a nested sequence of *nonempty compact* sets, then

$$\bigcap_{n=1}^{\infty} K_n \text{ _____}.$$



Proof: The proof is left as an Exercise. Write down a formal argument on your own.

Note: Closedness is *not* enough in Theorem 2, we need the compactness.

Consider $F_n = [n, \infty)$. Each F_n is closed, and $F_1 \supseteq F_2 \supseteq \dots$.

What is $\bigcap_{n=1}^{\infty} F_n$? _____

– Is the conclusion of Theorem 2 satisfied? _____

– What hypothesis of Theorem 2 does the family $\{F_n\}$ fail?

Open Covers and the Heine–Borel Theorem:

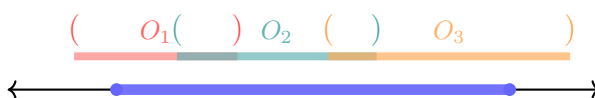
There is a third, and in many ways most powerful, way to think about compactness. It uses the concept of *open covers*.

Definition 3. Let $A \subseteq \mathbb{R}$. An **open cover** for A is a (possibly infinite) collection of open sets $\{O_\lambda : \lambda \in \Lambda\}$ such that

$$A \subseteq \bigcup_{\lambda \in \Lambda} O_\lambda.$$

A **finite subcover** is a finite subcollection $\{O_{\lambda_1}, \dots, O_{\lambda_n}\}$ from the original cover whose union still contains A .

Think of an open cover as a family of “patches” that together cover A . The question is: can we always get away with only finitely many patches?



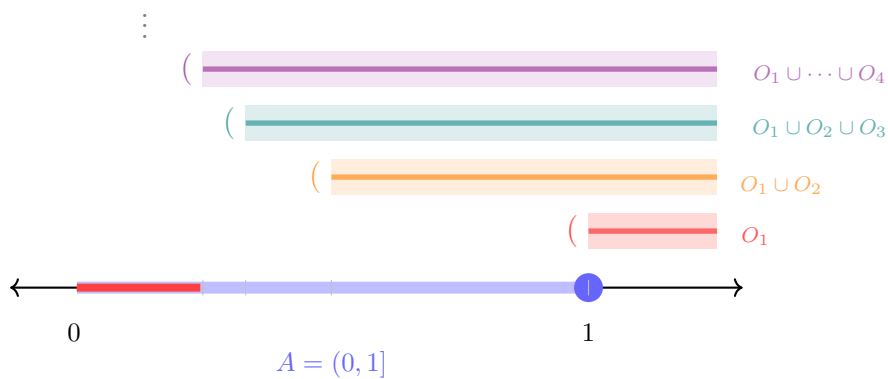
Example: Let $A = (0, 1]$ and for each $n \in \mathbb{N}$, define

$$O_n = \left(\frac{1}{n}, 2\right).$$

(a) Show that $\{O_n : n \in \mathbb{N}\}$ is an open cover of $(0, 1]$.

(b) Let $\{O_{n_1}, \dots, O_{n_k}\}$ be any finite subcollection of $\{O_n : n \in \mathbb{N}\}$. Can this be a subcover for A ?

(c) From your previous knowledge, is the interval $(0, 1]$ **compact**? Why or why not?



We are now ready for one of the most important theorems in analysis:

Theorem 3 (Heine–Borel Theorem). Let $K \subseteq \mathbb{R}$. The following are equivalent:

- (i) K is **compact**: every sequence in K has a subsequence converging to a limit in K .
- (ii) K is **closed and bounded**.
- (iii) K has the **finite cover property**: every open cover of K has a finite subcover.

The equivalence of (i) and (ii) was established in Theorem 1. Let us examine the implications (iii) \Rightarrow (ii) and (ii) \Rightarrow (iii).

Proof that (iii) \Rightarrow (ii): Suppose K has the finite cover property, that is, every open cover of K has a finite subcover.

Show that K is bounded. We will show the boundedness in steps.

Step 1: For each $x \in K$, let $O_x = V_1(x) = (x - 1, x + 1)$. Explain why $\{O_x : x \in K\}$ is an open cover of K .

Step 2: By (iii), there is a finite subcover $\{O_{x_1}, \dots, O_{x_n}\}$. Prove that this implies K is bounded.

Show that K is closed: We will show the closedness in steps.

Step 1: Suppose $(y_n) \subseteq K$ with $y_n \rightarrow y$. Prove that $y \in K$. Assume for contradiction $y \notin K$. For each $x \in K$, let $O_x = V_{|x-y|/2}(x)$. Prove that $\{O_x : x \in K\}$ is an open cover for K .

By (iii), there is a finite subcover $\{O_{x_1}, \dots, O_{x_n}\}$. Let

$$\varepsilon_0 = \min \left\{ \frac{|x_i - y|}{2} : 1 \leq i \leq n \right\}$$

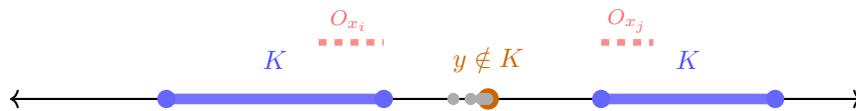
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Step 2: Write the $\varepsilon - N$ definition of $y_n \rightarrow y$.

For $N \in \mathbb{N}$, Show that $y_N \notin \bigcup_{i=1}^n O_{x_i}$.

Does it produce any contradiction?

Have we proved all the implications required for the proof?



Your turn:: For each of the following sets K , determine whether K is compact and cite appropriate Theorem or other resources. If K is not compact, then produce an open cover with no finite subcover.

(a) $K = [-5, 5]$: _____

(b) $K = \mathbb{N}$: _____.

(c) $K = \{x \in \mathbb{R} : |x^2 - 3| \leq 1\} = [-2, -\sqrt{2}] \cup [\sqrt{2}, 2]$: Compact? _____

Activity:

The following Exercise shows power of compactness and connect it to the concepts we studied earlier this semester.

Exercise: Prove that if K is compact and nonempty, then $\sup K$ exists *and* $\sup K \in K$.

Proof: The following hints will help you prove the result:

1 Why does $\sup K$ exist?

2 Use the definition of supremum to justify: for each $n \in \mathbb{N}$ there exists $a_n \in K$ with

$$\sup K - \frac{1}{n} < a_n \leq \sup K.$$

What does the Squeeze Theorem tell you about (a_n) ?

3 Use compactness of K to conclude $\sup K \in K$.

Follow-up question: Does $\sup(0, 1)$ belong to $(0, 1)$? _____ Why does the argument break down for open sets?

Summary: Three equivalent definitions of compactness in \mathbb{R}

