

Perfect Sets

In the previous section we characterized compactness as being *closed* and *bounded*. We now introduce another important property of sets in \mathbb{R} : **perfectness**.

Question: Are all closed sets intervals?

Two ways a closed set can “fail to be an interval.”

Recall that a point $x \in A$ is an **isolated point** if some ε -neighbourhood $V_\varepsilon(x)$ contains *no other point* of A ; equivalently, x is *not* a limit point of A .

Consider these four sets. For each one, mark whether each point is a limit point (L) or an isolated point (I) of the set.

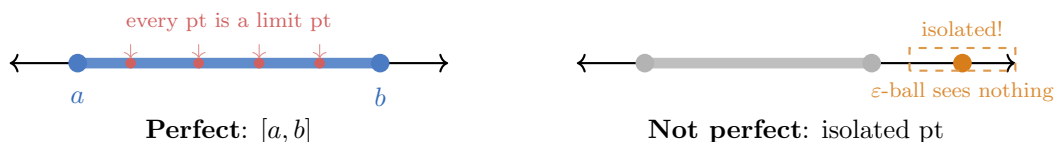
Set A	Sample point	L or I?	Closed?
$[0, 1]$	any $x \in (0, 1)$	_____	_____
$[0, 1]$	the endpoint $x = 0$	_____	_____
$\mathbb{N} = \{1, 2, 3, \dots\}$	any $n \in \mathbb{N}$	_____	_____
$\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$	$x = \frac{1}{n}$ for fixed n	_____	_____
$\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$	$x = 0$	_____	_____

Observation. Closed sets come in (at least) two flavours:

- **Interval-like:** $[0, 1]$ is closed and every point is a limit point.
- **Discrete-like:** \mathbb{N} is closed but *every* point is isolated.
- **Mixed:** $\{1/n\} \cup \{0\}$ has isolated points ($1/n$) and one limit point (0).

Question: What if we insist that a closed set has *no* isolated points at all? Then every point must be a limit point. This is exactly what compactness cannot guarantee on its own, and it leads to a new concept.

Definition 1. A set $P \subseteq \mathbb{R}$ is **perfect** if it is closed and contains *no* isolated points, i.e., every point of P is a limit point of P .



Your turn: For each set, decide whether it is perfect. If not, identify *precisely* which condition fails: not closed, has an isolated point, or both. Provide a brief justification.

(a) $[0, 1]$ Perfect? _____

(b) $\{0\} \cup \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$ Perfect? _____

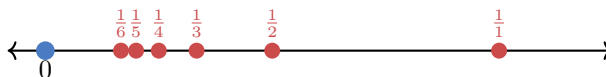
(c) \mathbb{N} Perfect? _____

(d) $[-1, 0) \cup (0, 1]$ Perfect? _____

What does “No Isolated Points” really mean?

The definition says every point of P is a limit point of P . Let us unpack this with a concrete family of examples that shows how *delicate* this condition is.

Example. Let $A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$ and $B = A \cup \{0\}$.



(a) Is A closed? _____

(b) Is $B = A \cup \{0\}$ closed? _____

(c) Is the point $x = 0$ an isolated point of B ? _____

(d) Is the point $x = \frac{1}{n}$ (for fixed n) an isolated point of B ? _____

(e) Is B perfect? _____ What single element would you need to add to (or remove from) B to move *closer* to a perfect set?

Observation: Even adding the “right” limit point 0 does not cure the isolated-point problem—the isolated points $\frac{1}{n}$ remain. The simplest perfect sets are closed intervals.

Theorem 1. Every closed interval $[a, b]$ with $a < b$ is perfect.

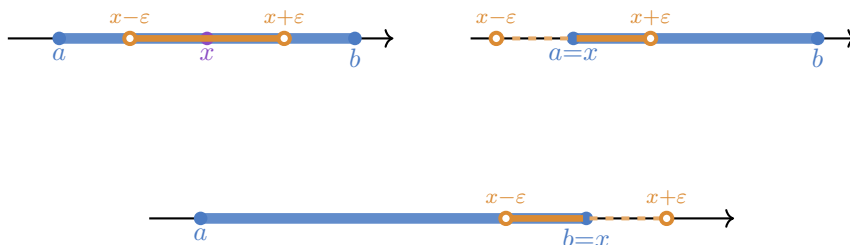
Proof sketch. We must verify two things: **(i)** $[a, b]$ is closed, and **(ii)** every $x \in [a, b]$ is a limit point of $[a, b]$.

Part (i): Why is $[a, b]$ closed?

Part (ii): Let $x \in [a, b]$ and let $\varepsilon > 0$ be arbitrary.

Step 1. Write down what $V_\varepsilon(x) \cap [a, b]$ looks like explicitly:

$$V_\varepsilon(x) \cap [a, b] = (\text{_____}, \text{_____}).$$



Step 2. Show that this intersection is a *nonempty open interval*.

Step 3. Explain why a nonempty open interval cannot be a singleton $\{x\}$, and therefore must contain a point $y \neq x$.

Conclude: $V_\varepsilon(x) \cap [a, b]$ contains a point $y \in [a, b]$ with $y \neq x$. Since $\varepsilon > 0$ was arbitrary, x is a limit point of $[a, b]$. Therefore, the closed interval $[a, b]$ is perfect. □

A Shocking Theorem: Perfect Sets are Big

Theorem 2. A nonempty perfect set is uncountable.

Activity:

Exercise 1. Let P be a perfect set and K compact.

(a) Is $P \cap K$ always compact? Justify your answer. (*Hint:* Is the intersection of a closed set and a compact set compact?)

(b) Is $P \cap K$ always perfect? If not, give a counterexample.

Exercise 2. Prove that no finite nonempty set $F \subseteq \mathbb{R}$ is perfect.

Follow-up. Use this and Theorem 2 to give another proof that every perfect set is infinite.