Connected Sets

We have now characterized compactness and perfectness. We turn to a third property: **connectedness**. Informally, a connected set is one that is "all in one piece." Making this precise is more subtle than it sounds.

Question: Consider these two sets:

$$E_1 = (1, 2) \cup (2, 5) \qquad \text{and} \qquad E_2 = (1, 5).$$

Intuitively E_1 has a "gap" at $x = 2$ while E_2 does not. But can we make this precise using only the language of limit points and closures?

Let $A = (1, 2)$ and $B = (2, 5)$. Work through the following:

(i) Is $2 \in A$? ________ Is $2 \in B$? ________ So $A \cap B =$ __________.

(ii) Is 2 a limit point of $A = (1, 2)$? __________ So $\overline{A} =$ __________.

(iii) Does $\overline{A} \cap B = [1, 2] \cap (2, 5)$ equal $\emptyset$? __________

(iv) Does $A \cap \overline{B} = (1, 2) \cap [2, 5]$ equal $\emptyset$? __________

(v) Now try $C = (1, 2]$ and $D = (2, 5)$: does $\overline{C} \cap D = \emptyset$? ________ Is $C \cap \overline{D} = \emptyset$?________

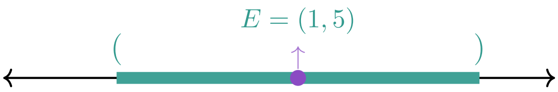
Observation: For $A = (1, 2)$ and $B = (2, 5)$: neither set contains a limit point of the other, even though they share the limit point 2. For $C = (1, 2]$ and $D = (2, 5)$: the closure of C hits D, so they are *not* separated in the same way. This "mutual closure-disjointness" is the right notion of separation.

Definition 1. Two nonempty sets $A, B \subseteq \mathbb{R}$ are **separated** if

$$\overline{A} \cap B = \emptyset \quad \text{and} \quad A \cap \overline{B} = \emptyset.$$

A set $E \subseteq \mathbb{R}$ is **disconnected** if it can be written as $E = A \cup B$ for some nonempty separated sets A and B. A set is **connected** if it is not disconnected.

Disconnected: $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$



Connected: no separation possible

Example 1: Let $A = (1, 2)$ and $B = (2, 5)$. Consider: $E = (1, 2) \cup (2, 5)$. Is the set E connected?

Example 2: Let $C = (1, 2]$ and $D = (2, 5)$, so that $C \cup D = (1, 5)$. Does the pair C, D give a separation of $(1, 5)$?

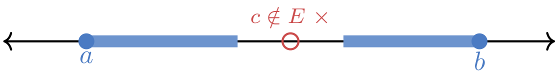
Characterisation of Connected Sets in $\mathbb{R}$

The surprising result is that in $\mathbb{R}$, connected sets are exactly intervals.

Theorem 1. A set $E \subseteq \mathbb{R}$ is connected if and only if: whenever $a, b \in E$ and $a < c < b$, we have $c \in E$.

In other words, E is connected $\iff$ E is an **interval**.

Connected: every in-between point present



Disconnected: gap at c

Proof sketch: **(⇒):** Suppose E is connected. Let $a, b \in E$ with $a < c < b$. Define:

$$A = (-\infty,\, c) \cap E \qquad \text{and} \qquad B = (c,\, \infty) \cap E.$$

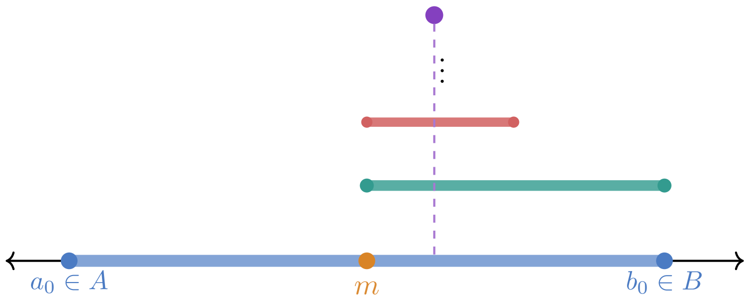
Step 1: Prove that $a \in A$ and $b \in B$.

Step 2. Prove that A and B are separated.

Step 3. Use the connectedness of E to prove that $c \in E$.

Therefore this direction is proved. □

($\Leftarrow$): Conversely, assume that E is an interval: whenever $a, b \in E$ satisfy $a < c < b$ for some c, then $c \in E$.



Let $E = A \cup B$ with A, B nonempty and disjoint. We show A and B are *not* separated.

Pick $a_0 \in A$, $b_0 \in B$, and assume WLOG $a_0 < b_0$.

Step 1. Why is $I_0 = [a_0, b_0] \subseteq E$?

Step 2. Bisect I_0 at its midpoint m. The midpoint lies in A or B. Explain why we can always choose a half $I_1 = [a_1, b_1] \subseteq I_0$ so that $a_1 \in A$ and $b_1 \in B$.

Step 3. Continuing, we get nested intervals $I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$ with:

- $a_n \in A$, $b_n \in B$ for each n, and
- $b_n - a_n = \dfrac{b_0 - a_0}{2^n} \to 0.$

By the **Nested Interval Property**, there exists ____________________.

Step 4. Show that $a_n \to x$ and $b_n \to x$.

Step 5. Prove that A and B are not separated.

Therefore, E is connected. $\square$

Your turn: For each set, decide whether it is connected. If not, exhibit an explicit separation $E = A \cup B$.

(a) $E = (-3, 0) \cup (0, 3)$ Connected? ____________

(b) $E = \mathbb{R} \setminus \{0\}$ Connected? ____________

(c) $E = [0, 1] \cup \{2\}$ Connected? ____________

(d) $E = [0, 1] \cup [1, 2]$ Connected? ____________