
Instructions: Show all work clearly and justify each conclusion. Collaboration is encouraged, but *write up your solutions individually in your own words*. For any **prove/disprove** problem: either give a proof, or give a specific counterexample (with a brief explanation of why it works).

Problem 1. Let $A, B \subseteq \mathbb{R}$ be nonempty and bounded.*

- (a) Prove that if $A \subseteq B$, then $\sup A \leq \sup B$.
- (b) **Prove or disprove:** If $\sup A < \inf B$, then there exists a real number c such that $a < c < b$ for all $a \in A$ and $b \in B$.

Problem 2. Let $a, b \in \mathbb{R}$. Recall the triangle inequality: $|x + y| \leq |x| + |y|$ for all $x, y \in \mathbb{R}$.

- (a) Use the identity $a = (a - b) + b$ and apply the triangle inequality to obtain an upper bound for $|a|$ in terms of $|a - b|$ and $|b|$.
- (b) Interchange the roles of a and b to obtain a similar inequality for $|b|$.
- (c) Deduce that $||a| - |b|| \leq |a - b|$.

Problem 3. Let E and F be nonempty subsets of \mathbb{R} bounded above, and define $E + F = \{x + y : x \in E \text{ and } y \in F\}$.

- (a) Let $\alpha = \sup E$ and $\beta = \sup F$. Show that $\alpha + \beta$ is an upper bound for $E + F$.
- (b) Let M be an arbitrary upper bound for $E + F$, and fix $x \in E$. Show that $\beta \leq M - x$.
- (c) Conclude that $\sup(E + F) = \alpha + \beta$.

Problem 4. For each set below, determine and **justify** $\sup A$ and $\inf A$, and state whether they belong to the set.

- (a) $A = \{x \in \mathbb{R} : x^2 < 2\}$
- (b) $A = \{1 - 1/n : n \in \mathbb{N}\}$

Problem 5. Recall that a set is called *countable* if it is finite or can be put into a one-to-one correspondence with \mathbb{N} .

- (a) Prove that any finite subset of \mathbb{R} is countable.
- (b) Show that $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ is countable by explicitly describing a bijection with \mathbb{N} .
- (c) Let $B = \mathbb{Z} \cap [0, \infty)$. Prove that B is countable.
- (d) Prove that the union of a finite set and a countable set is countable.

*A set is bounded if it is both bounded above and bounded below.