MATH-4630-01 | **Homework Assignment #2**
DUE: *Tuesday, Feb 17, 11:59p* | Sections 2.1–2.5

Instructions: Show all work clearly and justify each conclusion. Collaboration is encouraged, but *write up your solutions individually in your own words.* For any **prove/disprove** problem: either give a proof, or give a specific counterexample.

Problem 1. Using the definition of convergence, prove the following limits.

(a) $\lim_{n\to\infty} \left(\frac{3n+1}{2n+5}\right) = \frac{3}{2}$. (b) $\lim_{n\to\infty} \left(\frac{n}{n^2+1}\right) = 0$.

Problem 2. Give an example of each or state that it is impossible.

(a) A sequence with an infinite number of ones that does not converge to one.
(b) A sequence with an infinite number of ones that converges to a limit not equal to one.
(c) A divergent sequence such that for every $n \in \mathbb{N}$ it is possible to find n consecutive ones somewhere in the sequence.

Problem 3. Let $(a_n) \to 0$, and use the Algebraic Limit Theorem to compute each limit (assuming the fractions are always defined):

(a) $\lim_{n\to\infty} \left(\frac{1+2a_n}{1+3a_n-4a_n^2}\right)$ (b) $\lim_{n\to\infty} \left(\frac{(a_n+2)^2-4}{a_n}\right)$

Problem 4. Give an example of each of the following, or state that such a request is impossible by referencing the proper theorem(s):

(a) sequences (x_n) and (y_n), which both diverge, but whose sum $(x_n + y_n)$ converges;
(b) sequences (x_n) and (y_n), where (x_n) converges, (y_n) diverges, and $(x_n + y_n)$ converges;
(c) a convergent sequence (b_n) with $b_n \neq 0$ for all n such that $(1/b_n)$ diverges;
(d) an unbounded sequence (a_n) and a convergent sequence (b_n) with $(a_n - b_n)$ bounded;
(e) two sequences (a_n) and (b_n), where $(a_n b_n)$ and (a_n) converge but (b_n) does not.

Problem 5. Prove that the sequence defined by $x_1 := 1$ and $x_{n+1} := \sqrt{2+x_n}$ for all $n \geq 1$ is convergent. Find the limit.

Problem 6. Let (a_n) be a bounded sequence.

(a) Prove that the sequence defined by $y_n = \sup\{a_k : k \geq n\}$ converges.
(b) The *limit superior* of (a_n) is defined by $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} y_n$. Similarly, $z_n = \inf\{a_k : k \geq n\}$ and $\liminf_{n\to\infty} a_n = \lim_{n\to\infty} z_n$. Prove that (z_n) converges.
(c) Prove that $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$ for every bounded sequence, and give an example where the inequality is strict.
(d) Show that $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$ if and only if $\lim_{n\to\infty} a_n$ exists.

Problem 7. Give an example of each of the following, or argue that such a request is impossible.

(a) A sequence that has a subsequence that is bounded but contains no subsequence that converges.
(b) A sequence that does not contain 0 or 1 as a term but contains subsequences converging to each of these values.
(c) A sequence that contains subsequences converging to every point in $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\}$.
(d) A sequence that contains subsequences converging to every point in $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\}$, and no subsequences converging to points outside of this set.

Problem 8. Decide whether the following propositions are true or false, providing a short justification for each conclusion.
(a) If every proper subsequence of (x_n) converges, then (x_n) converges as well.
(b) If (x_n) contains a divergent subsequence, then (x_n) diverges.
(c) If (x_n) is bounded and diverges, then there exist two subsequences of (x_n) that converge to different limits.
(d) If (x_n) is monotone and contains a convergent subsequence, then (x_n) converges.

Problem 9 (**Bonus problem:**)**.** Suppose that every subsequence of a sequence (x_n) has a further subsequence that converges to 0. Prove that $\lim_{n\to\infty} x_n = 0$. (**Hint:** Prove using contradiction!)
