
Instructions: Show all work clearly and justify each conclusion. Collaboration is encouraged, but *write up your solutions individually in your own words.*

Problem 1. Decide whether each of the following series converges or diverges.

(a) $\sum_{n=1}^{\infty} \frac{1}{2^n + n}$

(b) $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$

(c) $1 - \frac{3}{4} + \frac{4}{6} - \frac{5}{8} + \frac{6}{10} - \dots$

(d) $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$

(e) $1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \frac{1}{5} - \frac{1}{6^2} + \dots$

Problem 2. Give an example of each or explain why the request is impossible referencing the proper theorem(s).

- (a) Two series $\sum x_n$ and $\sum y_n$ that both diverge but where $\sum x_n y_n$ converges.
 - (b) A convergent series $\sum x_n$ and a bounded sequence (y_n) such that $\sum x_n y_n$ diverges.
 - (c) Two sequences (x_n) and (y_n) where $\sum x_n$ and $\sum(x_n + y_n)$ both converge but $\sum y_n$ diverges.
 - (d) A sequence (x_n) satisfying $0 \leq x_n \leq 1/n$ where $\sum(-1)^n x_n$ diverges.
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Definition 1. A series $\sum_{n=1}^{\infty} a_n$ is said to **converge conditionally** if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Problem 3. Consider each of the following propositions. Provide short proofs for those that are true and counterexamples for any that are not.

- (a) If $\sum a_n$ converges absolutely, then $\sum a_n^2$ also converges absolutely.
 - (b) If $\sum a_n$ converges and (b_n) converges, then $\sum a_n b_n$ converges.
 - (c) If $\sum a_n$ converges conditionally, then $\sum n^2 a_n$ diverges.
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Problem 4. Decide whether the following statements are true or false.

- (a) An open set that contains every rational number must necessarily be all of \mathbb{R} .
 - (b) Every nonempty open set contains a rational number.
 - (c) Every bounded infinite closed set contains a rational number.
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Problem 5. Assume A is an open set and B is a closed set. Determine if the following sets are definitely open, definitely closed, both, or neither.

- (a) $\overline{A \cup B}$
- (b) $A \setminus B = \{x \in A : x \notin B\}$
- (c) $(A^c \cup B)^c$

(d) $(A \cap B) \cup (A^c \cap B)$

(e) $\overline{A}^c \cap \overline{A}^c$

Problem 6.(a) Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.(b) Does this result about closures extend to infinite unions of sets?
