

Instructions: Show all work clearly and justify each conclusion. Collaboration is encouraged, but *write up your solutions individually in your own words.*

For any **prove/disprove** problem: either give a proof, or give a specific counterexample (with a brief explanation of why it works).

Problem 1. Provide an example of each or explain why the it is impossible.

- (a) Two functions f and g , neither of which is continuous at 0, but such that $f(x)g(x)$ and $f(x) + g(x)$ are continuous at 0.
- (b) A function $f(x)$ continuous at 0 and a function $g(x)$ not continuous at 0 such that $f(x) + g(x)$ is continuous at 0.
- (c) A function $f(x)$ continuous at 0 and a function $g(x)$ not continuous at 0 such that $f(x)g(x)$ is continuous at 0.
- (d) A function $f(x)$ not continuous at 0 such that $[f(x)]^2$ is continuous at 0.

Problem 2. Assume $g : \mathbb{R} \rightarrow \mathbb{R}$ is continuous on \mathbb{R} and let

$$K = \{x \in \mathbb{R} : g(x) = 0\}.$$

Show that K is a closed set.

Problem 3. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous at c and suppose $f(c) > 0$. Show that there exists a neighborhood $V_\delta(c)$ of c such that if $x \in V_\delta(c)$, then $f(x) > 0$. (**Hint:** Use the ε - δ definition of continuity of f at c with $\varepsilon = \frac{f(c)}{2}$.)

Problem 4. Consider the function $f : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ defined by

$$f(x) = \frac{1}{x^2}.$$

- (a) Show that f is uniformly continuous on $[1, \infty)$.
- (b) Show that f is not uniformly continuous on $(0, 1]$.

Problem 5. Let $f : A \rightarrow \mathbb{R}$ be a function such that there exists a constant $M > 0$ with

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq M \quad \text{for all } x \neq y.$$

Show that if $f : A \rightarrow \mathbb{R}$ is Lipschitz, then f is uniformly continuous on A .

Problem 6. Provide an example of each of the following, or explain why it is impossible.

- (a) A continuous function defined on an open interval with range equal to a closed interval.
- (b) A continuous function defined on a closed interval with range equal to an open interval.
- (c) A continuous function defined on an open interval with range equal to an unbounded closed set different from \mathbb{R} .
- (d) A continuous function defined on all of \mathbb{R} with range equal to \mathbb{Q} .

Problem 7. Let f be continuous on the interval $[0, 1]$ to \mathbb{R} and suppose that $f(0) = f(1)$. Prove that there exists a point $c \in [0, \frac{1}{2}]$ such that

$$f(c) = f(c + \frac{1}{2}).$$

This result shows that, at any given time, there exist antipodal points on the Earth's equator that have the same temperature.
