Hermitian projections on some Operator spaces

Something old, Something new, Something borrowed, something blue

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New Faces in Operator Theory & Function Theory
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Outline

Introduction

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- Projections are operators with a simple spectral structure and easy to understand.
- Projections are the building blocks of many other operators (i.e., Spectral theorem for compact self-adjoint operator).
- Hermitian operators emerged as a generalization of the self-adjoint operators on Hilbert spaces.
- Hermitian + Projection = Hermitian projection.

Notation

- X, Y denote complex Banach spaces.
- $\mathcal{B}(X, Y)$ denotes the space of all bounded linear operators from X to Y.
- $\mathcal{H}$, $\mathcal{K}$ denote Hilbert spaces.
- For $\mathcal{H} = \mathbb{C}^n$, $\mathcal{S}(\mathcal{H})$ and $\mathcal{A}(\mathcal{H})$ denote the space of symmetric and anti-symmetric matrices respectively.

Definitions & Examples

Definition (Isometry)

Given two normed spaces X and Y, a linear map $T : X \to Y$ is an isometry if $\|Tx\| = \|x\|$ for every $x \in X$.

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Example

For $\lambda \in \mathbb{R}$, the map $Tx = \lambda x$ is a Hermitian operator (trivial Hermitian operator).

Spaces with Trivial Hermitian Operators

Examples (Berkson, Sourour, Studia Math., 1974)

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- $\mathrm{Lip}[0,1]$ with $\|f\| = \|f\|_\infty + \operatorname{ess\,sup} |f'|$.
- The Hardy space $H^p(\mathbb{D})$, $1 \leq p \leq \infty$, $p \neq 2$.

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Let $C[0,1]$ be the space of complex valued continuous functions on $[0,1]$ with the maximum norm. Consider $H(f)(x) = f(1-x)$. Choose $f(x) = x$:

$$e^{2\pi itH}(f)(x) = (\cos 2\pi t)\, x + i \sin 2\pi t\, (1-x), \qquad t \in \mathbb{R},$$

which implies

$$\|e^{2\pi itH}(f)\| = \max_{x\in[0,1]} \sqrt{x^2 - (2x-1)\sin^2(2\pi t)}.$$

For $t = \frac{1}{8}$, we have $\|e^{2\pi itH}(f)\| = \frac{\sqrt{2}}{2} \neq \|f\|$. Hence $e^{2\pi i \frac{1}{8} H}$ is not an isometry.

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Theorem (J. Jamison, Linear Algebra Appl., 2007)

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Example

Every orthogonal projection on a complex Hilbert space is a Hermitian projection.

Example (Trivial Hermitian projections)

- $C^1[0, 1]$.
- $\mathrm{AC}[0, 1]$.
- $\mathrm{Lip}[0, 1]$.
- $H^p(\mathbb{D})$, $p \neq 2$.

Hermitian Projections on Operator Spaces

Theorem (Stachó and Zalar, Linear Algebra Appl., 2004)

1. *Let* $P : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ *be a Hermitian projection. Then* P *has the form* $x \mapsto Qx$ *or* $x \mapsto xQ$ *for some* $Q = Q^* = Q^2 \in \mathcal{B}(\mathcal{H})$.
2. *Let* $P : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H})$ *be a Hermitian projection. Then either* $P = 0$ *or* $P = I$.
3. *Let* $P : \mathcal{A}(\mathcal{H}) \to \mathcal{A}(\mathcal{H})$ *be a Hermitian projection. Then there exists a unit vector* $\alpha \in \mathcal{H}$ *such that either* $Px = Qx + xQ^T$ *or* $(I - P)(x) = Qx + xQ^T$ *where* $Q = \alpha \otimes \alpha$.

Hermitian Projections on Vector-Valued Continuous Functions

Proposition (Dey, Botelho, Ilišević)

Let K be a compact Hausdorff space, E a Banach space, and T a Hermitian projection on $C(K, E)$. Then there exists a map $t \mapsto A(t)$ from K into $\mathcal{HP}(E)$ such that

$$TF(t) = A(t)F(t), \quad \text{for every } F \in C(K, E) \text{ and } t \in K.$$

Where $\mathcal{HP}(E)$ denotes the space of all Hermitian projections on E.

Hermitian Projections on $L^p(\Omega, X)$

Proposition (Dey, Botelho, Ilišević)

Let (Ω, Σ, μ) and X denote a finite measure space and a separable Banach space respectively. If T is a Hermitian operator on $L^p(\Omega, X)$, $1 \leq p < \infty$, $p \neq 2$, then T is a projection if and only if there exists a strongly measurable function $t \mapsto A(t)$ from Ω into $\mathcal{HP}(X)$ such that

$$(Tf)(t) = A(t)f(t), \quad \text{for every } f \in L^p(\Omega, X) \text{ and } t \in \Omega.$$

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Corollary

Let E be a complex linear symmetric sequence space different from ℓ^2. Then T on E is a Hermitian projection if and only if there exists $a = (a(k)) \in E$ with all entries equal to 0 or 1, such that $Tx = ax = (a(k)x(k))$.

The Main Question

For two Banach spaces X and Y, what are the Hermitian projections on $\mathcal{B}(X, Y)$?

Ideal Pair of Banach Spaces

Definition (Khalil, Saleh, 2003)

A pair of Banach spaces (X, Y) is called an **ideal pair** if:

1. X and Y are reflexive.
2. X and Y^* are strictly convex.
3. X^* has the approximation property.
4. $\mathcal{K}(X, Y)$ is an M-ideal in $\mathcal{B}(X, Y)$.

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Example (Cohen, 1973)

(ℓ^p, ℓ^q) is an ideal pair for $1 < p \leq q < \infty$.

Hermitian Projections on $\mathcal{B}(X, Y)$

Theorem (Dey, Botelho, Ilišević)

Let (X, Y) be an ideal pair such that X is not isometric to Y^*. Then $P : \mathcal{B}(X, Y) \to \mathcal{B}(X, Y)$ is a Hermitian projection if and only if:

1. *Either there exists a Hermitian projection $Q \in \mathcal{B}(X)$ such that $P(A) = AQ$ for every $A \in \mathcal{B}(X, Y)$, or*
2. *there exists a Hermitian projection $R \in \mathcal{B}(Y)$ such that $P(A) = RA$ for every $A \in \mathcal{B}(X, Y)$.*

Two Key Ingredients of the Proof

Theorem (Fong, Sourour, C.J. Math, 1979)

If $\sum_{k=1}^{m} A_k T B_k = 0$ for all $T \in \mathcal{B}(X, Y)$, then either $\{B_1, \ldots, B_m\}$ is linearly independent and $A_i = 0$ for every i, or $\{B_1, \ldots, B_m\}$ is linearly dependent.

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Theorem (Khalil, Saleh, Proc. AMS, 2003)

Let (X, Y) be an ideal pair such that X and Y^* are not isometric. Then $T : \mathcal{B}(X, Y) \to \mathcal{B}(X, Y)$ is a surjective isometry if and only if there are surjective isometries $U \in \mathcal{B}(Y)$ and $V \in \mathcal{B}(X)$ such that $T(f) = UfV$ for all $f \in \mathcal{B}(X, Y)$.

Sketch of the Proof

1. Given a Hermitian projection T and $t \in (0,1)$, we get

$$Tf = \frac{U_t f V_t - e^{it} f}{1 - e^{it}}$$

with U_t and V_t surjective linear isometries on Y and X respectively.

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2. Since T is a projection, for each $t \in (0, 1)$:
$$U_t^2 f V_t^2 - (1 + e^{2\pi it}) U_t f V_t + e^{2\pi it} f = 0, \quad \text{for every } f \in \mathcal{B}(X, Y).$$

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Remark

If X and Y^* are isometric, then a surjective isometry T may have the form $T(f) = Uf^*V$, which leads to no projection.

A Nice Corollary

Corollary

Let $\mathcal{H}$, $\mathcal{K}$ be two non-isometric Hilbert spaces. Then $P : \mathcal{B}(\mathcal{H}, \mathcal{K}) \to \mathcal{B}(\mathcal{H}, \mathcal{K})$ is a Hermitian projection if and only if one of the following holds:

1. *There exists a Hermitian projection $Q \in \mathcal{B}(\mathcal{H})$ such that $P(A) = AQ$ for every $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, or*
2. *there exists a Hermitian projection $R \in \mathcal{B}(\mathcal{K})$ such that $P(A) = RA$ for every $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$.*

Thank You!

Questions?

Hermitian projections on some Banach spaces and related topics

Botelho, Dey, Ilišević — Linear Algebra and Its Applications, 2020