

Projections in the Algebra Generated by an n -Potent Operator

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Outline

- 1 Motivation
- 2 Setup and Spectral Preliminaries
- 3 Main Results
- 4 Illustrative Example
- 5 Open Questions

Two Threads in the Literature

Thread 1: Projections in the convex hull of isometries.

- **Botelho** (JMAA 2008): $\lambda T_1 + (1-\lambda)T_2$ a projection on $C(\Omega) \Rightarrow \lambda = \frac{1}{2}$ forced; always a generalized bi-circular projection.
- **Hosseini–Jiménez-Vargas** (*Bull. Malays. Math. Sci. Soc.*, 2021): the unique projection in $\text{co}\{U_1, U_2, U_3\}$ is $\frac{1}{3}(U_1 + U_2 + U_3)$.
- **Botelho–Dey–Easley** (JMAA 2023): if U is a periodic surjective isometry with $U^m = I$, the unique projection in $\text{co}\{U^k\}_{k=0}^{m-1}$ is $\frac{1}{m} \sum_{k=0}^{m-1} U^k$.

Pattern: the convex hull yields at most **one** projection.

Thread 2: Projections from a single operator.

- **Bikchentaev–Yakushev** (LAA 2011): for $T^3 = T$, characterized the projections in $\text{span}\{T, T^2\}$ – the $n = 3$ case of our theorem.
- **Ilišević–Li–Poon** (JMAA 2022): when is $T = \lambda_1 P + \lambda_2(I-P)$ an isometry?

The Gap and Our Contribution

What is missing

For general n -potent T ($T^n = T$, $n \geq 2$), the projections lying in the algebra

$$\text{comb}(T) = \text{span}\{T, T^2, \dots, T^{n-1}\}$$

have **not been systematically characterized**.

Note: $I \notin \text{comb}(T)$ in general; $T^{n-1} = I \Leftrightarrow T$ is periodic. We work in the subalgebra without the identity.

This talk

We give a *complete* classification:

- Every projection in $\text{comb}(T)$ is a sum of Riesz projections, uniquely indexed by a subset $S \subseteq \sigma(T) \setminus \{0\}$, via *explicit, computable formulas* – no contour integration required.
- The full family forms a **Boolean algebra** isomorphic to $\mathcal{P}(\sigma_0(T))$, with exactly $2^{|\sigma_0(T)|}$ elements.

Why the Algebra, Not Just the Convex Hull?

Convex hull $\text{co}\{T^k\}$

Asks: which *averages*

$$\sum_{k=0}^{m-1} c_k T^k, \quad c_k \geq 0, \quad \sum c_k = 1$$

are projections?

Answer (when $T^m = I$): *at most one* – the Cesàro mean

$$A = \frac{1}{m} \sum_{k=0}^{m-1} T^k,$$

the projection onto $\text{Fix}(T)$.

Algebra $\text{comb}(T)$

Asks: which *arbitrary linear combinations*

$$\sum_{k=0}^{n-1} a_k T^k, \quad a_k \in \mathbb{C}$$

are projections?

Answer (this paper): *exactly* $2^{|\sigma_0(T)|}$ projections – a complete **Boolean algebra** indexed by subsets of $\sigma(T) \setminus \{0\}$.

Important:

The algebra captures the *full spectral structure* of T , not just its averaging behavior.

n -Potent Operators and Their Spectrum

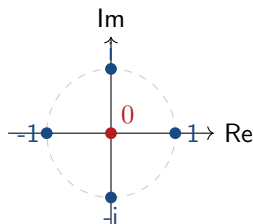
Definition

$T \in B(X)$ is **n -potent** if $T^n = T$ ($n \geq 2$).

Examples: projections ($n = 2$), tripotents ($n = 3$), periodic operators $T^m = I$ ($n = m + 1$).

Proposition

If T is n -potent then $\sigma(T) \subseteq \{0, 1, \omega, \dots, \omega^{n-2}\}$ where $\omega = e^{2\pi i/(n-1)}$, and every eigenvalue is *semisimple*.



$$\sigma(T) \subseteq \{0, 1, i, -1, -i\} \text{ for } n = 5$$

Riesz Projections: The Key Tool

Definition

For $\lambda \in \sigma(T)$ isolated, the **Riesz projection** is

$$P_{\{\lambda\}} := -\frac{1}{2\pi i} \oint_{\Gamma_\lambda} (T - zI)^{-1} dz.$$

For n -potent T , the Riesz projections satisfy:

- $P_{\{\lambda\}}^2 = P_{\{\lambda\}}$, $P_{\{\lambda\}}T = TP_{\{\lambda\}}$.
- $P_{\{\lambda\}}P_{\{\mu\}} = 0$ for $\lambda \neq \mu$, $\sum_{\lambda \in \sigma(T)} P_{\{\lambda\}} = I$.
- $\text{Ran}(P_{\{\lambda\}}) = \text{Ker}(T - \lambda I)$ (semisimplicity is essential here)

Spectral decomposition

T is n -potent if and only if $T = \sum_{\lambda \in \sigma(T)} \lambda P_{\{\lambda\}}$, where $\{P_{\{\lambda\}}\}$ is a *unique* finite resolution of identity.

Explicit Formula for Riesz Projections

Lemma (Resolvent expansion)

$$\text{For } z \in \rho(T): (zI - T)^{-1} = \frac{I}{z} + \frac{T^{n-1}}{z(z^{n-1} - 1)} + \sum_{k=1}^{n-2} \frac{z^{n-k-2}}{z^{n-1} - 1} T^k.$$

Theorem

With $\omega = e^{2\pi i/(n-1)}$, for each $j = 0, \dots, n-2$:

$$\omega^j \in \sigma(T) \iff P_{\{\omega^j\}} = \frac{1}{n-1} \sum_{k=1}^{n-1} \omega^{-jk} T^k$$

is a nonzero projection satisfying $TP_{\{\omega^j\}} = \omega^j P_{\{\omega^j\}}$.

Consequences of the Explicit Formula

DFT structure

The formula

$$P_{\{\omega^j\}} = \frac{1}{n-1} \sum_{k=1}^{n-1} \omega^{-jk} T^k$$

is a **discrete Fourier transform** of $\{T, T^2, \dots, T^{n-1}\}$ in \mathbb{Z}_{n-1} .

No contour integration needed in practice.

Theorem

T^{n-1} is itself a projection:

$$T^{n-1} = \sum_{\lambda \neq 0} P_{\{\lambda\}}.$$

Geometrically: T^{n-1} projects onto $\text{Ran}(T)$ along $\ker(T)$.

Moreover,

$$T \text{ periodic} \iff T^{n-1} = I.$$

Main Theorem: Classification of Projections in $\text{comb}(T)$

Theorem

Let T be n -potent and $P = \sum_{i=1}^k a_i T^i \in \text{comb}(T)$ a projection. Then P has a *unique* decomposition

$$P = \sum_{\lambda \in \sigma(T)} \beta_\lambda P_{\{\lambda\}}, \quad \beta_\lambda = \sum_{i=1}^{n-1} a_i \lambda^i.$$

Moreover, $\beta_\lambda \in \{0, 1\}$ for every λ , and $\text{Ran}(P) = \bigoplus_{\beta_\lambda=1} P_{\{\lambda\}}(X)$.

Converse: For every $S \subseteq \sigma_0(T)$, the operator $P_S := \sum_{\lambda \in S} P_{\{\lambda\}}$ lies in $\text{comb}(T)$ and is a projection.

Bottom line

$$\{\text{projections in } \text{comb}(T)\} \longleftrightarrow \mathcal{P}(\sigma_0(T)).$$

Boolean Algebra of Projections

Write $\sigma_0(T) = \sigma(T) \setminus \{0\}$ and $P_S := \sum_{\lambda \in S} P_{\{\lambda\}}$ for $S \subseteq \sigma_0(T)$.

Corollary

- Every projection in $\text{comb}(T)$ equals P_S for a *unique* $S \subseteq \sigma_0(T)$.
- $S \mapsto P_S$ is a bijection $\mathcal{P}(\sigma_0(T)) \longleftrightarrow \{\text{projections in } \text{comb}(T)\}$.
- $\{P_S\}$ is a **Boolean algebra** of projections:

$$P_S \vee P_R = P_{S \cup R}, \quad P_S \wedge P_R = P_{S \cap R}, \quad P_S^c = P_{\sigma_0(T) \setminus S},$$

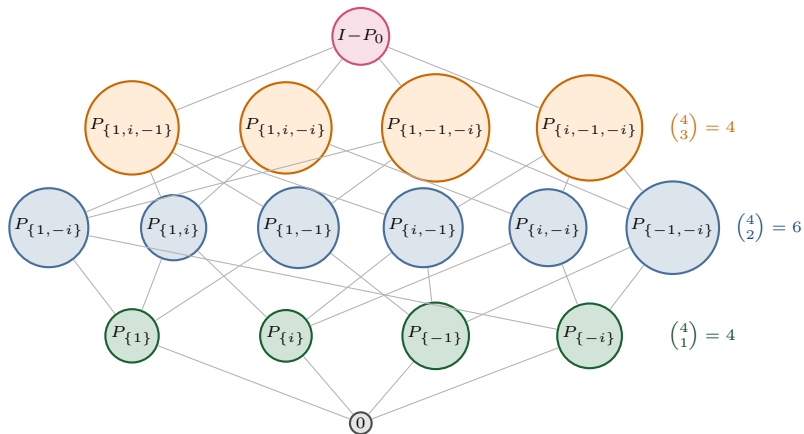
isomorphic to $(\mathcal{P}(\sigma_0(T)), \cup, \cap, ^c)$.

Count

Exactly $2^{|\sigma_0(T)|}$ projections in $\text{comb}(T)$. For $n = 5$ with full spectrum:
 $2^4 = \mathbf{16}$ projections.

The Boolean Lattice: $n = 5$ Case

$\sigma_0(T) \subseteq \{1, i, -1, -i\}$, $|\mathcal{P}(\sigma_0(T))| = 2^4 = 16$ projections.



Example: 5-Potent Operator

Let T be 5-potent: $\sigma(T) \subseteq \{0, 1, i, -1, -i\}$, $\omega = i$.

Riesz projections:

$$P_{\{i\}} = \frac{1}{4}(-iT - T^2 + iT^3 + T^4), \quad P_{\{-1\}} = \frac{1}{4}(-T + T^2 - T^3 + T^4).$$

Projection for $S = \{i, -1\}$:

$$P_{\{i, -1\}} = P_{\{i\}} + P_{\{-1\}} = \left(-\frac{1+i}{4}\right)T + \left(\frac{-1+i}{4}\right)T^3 + \frac{1}{2}T^4.$$

Recovery: Given $P = \left(-\frac{1+i}{4}\right)T + \left(\frac{-1+i}{4}\right)T^3 + \frac{1}{2}T^4$, compute

$$\beta_\lambda = \sum_j a_j \lambda^j:$$

$$\beta_0 = 0, \beta_1 = 0, \beta_i = 1, \beta_{-1} = 1, \beta_{-i} = 0 \implies P = P_{\{i\}} + P_{\{-1\}}. \checkmark$$

Connection to tripotents ($n = 3$)

For $n = 3$: $\sigma_0(T) \subseteq \{1, -1\}$ and $T = P_{\{1\}} - P_{\{-1\}}$, recovering

Bikchentaev–Yakushev (2011) exactly.

- 1 **Hermitian Riesz projections:** When are the $P_{\{\lambda\}}$ bicircular (Hermitian)? On Hilbert spaces, when are they orthogonal?
- 2 **Convex hull vs. algebra:** Which $P_S \in \text{comb}(T)$ are also in $\text{co}\{T^k\}$? Characterize the intersection.
- 3 **C^* -algebra extension:** Classify projections in $\text{Alg}(T, T^*)$ on Hilbert spaces.

Thank You!



Questions?

Projections in the Algebra Generated by an n -Potent Operator

arxiv.org/abs/2512.22497 [math.FA] December 2025