Projections in the Combination of Operators of Finite Orders

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New Faces in Operator Theory & Function Theory
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Outline

Why Study Projections and Isometries?

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- Isometries and projections are interconnected.

Definitions & Examples

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Definition (Reflection operator)

An operator $R\colon X \to X$ is a **reflection** if $R^2 = I$.

Creating Projections from Reflections

Example

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An Example in the Infinite-Dimensional World

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Remark

In general, if R is an isometry and a reflection, then $P = \frac{1}{2}(I + R)$ is a projection.

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We ask:

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Consider $P\colon \mathbb{R}^3 \to \mathbb{R}^3$ (with $\|\cdot\|_\infty$) defined by $P(x_1, x_2, x_3) = (x_1 + x_2, 0, x_3)$. P is a projection. If $P = \frac{1}{2}(I+R)$, then $R(x_1, x_2, x_3) = (x_1 + 2x_2, -x_2, x_3)$, which is not an isometry — contradiction.

Projections as a Combination of Two Isometries

(Botelho–Dey–Easley, 2023) For two isometries T_0 and T_1 on a Banach space X, consider $Q = \lambda_0 T_0 + \lambda_1 T_1$ $(Q \neq I)$ with $\lambda_0, \lambda_1 > 0$ and $\lambda_0 + \lambda_1 = 1$. If Q is a projection, then $\lambda_0 = \lambda_1 = \frac{1}{2}$.

Projections in the Convex Hull of Operators of Finite Order

(Botelho–Dey–Easley, 2023) For an operator T of order n (i.e., $T^n = I$), and positive scalars $\lambda_0, \lambda_1, \dots, \lambda_{n-1}$ with $\sum \lambda_i = 1$, consider

$$Q = \lambda_0 I + \lambda_1 T + \lambda_2 T^2 + \cdots + \lambda_{n-1} T^{n-1}.$$

Then Q is a projection if and only if $\lambda_0 = \lambda_1 = \cdots = \lambda_{n-1} = \frac{1}{n}$.

Future Work: The n-Potent Case

- For an operator T with $T^n = T$, and scalars $a_1, a_2, \ldots, a_{n-1}$, consider

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For what values of the scalars $a_1, a_2, \ldots, a_{n-1}$ is Q a projection?

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Definition

An operator T is said to be n-**potent** if $T^n = T$.

For $n = 2$ and Beyond — Ongoing Project

For $n = 2$, let $T^2 = T$. Then $Q = a_1 T$ is a projection if and only if $a_1 = 0$ or $a_1 = 1$. The only projections are $Q = 0$ or $Q = T$.

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$n = 2$	Comments
0	"trivial projection"
T	"basic projection"

For $n = 3$

For $n = 3$, let $T^3 = T$. Then $Q = a_1 T + a_2 T^2$ is a projection if and only if $Q \in \left\{0,\ T^2,\ \frac{T + T^2}{2},\ \frac{-T + T^2}{2}\right\}$.

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$n = 3$	Comments
0	"trivial projection"
$\dfrac{T + T^2}{2}$, $\dfrac{-T + T^2}{2}$	"basic projections"
T^2	sum of basic projections

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Let $T^{k+1} = T$ and let λ be a k-th root of unity. The projection

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For n

#	Expression	Type
1	0	trivial
2	$\frac{1}{3}(T + T^2 + T^3)$	basic
3	$\frac{1}{3}(\omega T + \omega^2 T^2 + T^3)$	basic
4	$\frac{1}{3}(\omega^2 T + \omega T^2 + T^3)$	basic
5	$\alpha T + \beta T^2 + \frac{2}{3} T^3$	#2+#3
6	$\beta T + \alpha T^2 + \frac{2}{3} T^3$	#2+#4
7	$\frac{1}{3}(-T - T^2 + 2T^3)$	#3+#4
8	T^3	[illegible]

The General Open Question (Now Resolved!)

Open question at the time

For an n-potent operator T (i.e., $T^n = T$), is it possible to classify all projections in $\text{span}\{T, T^2, \ldots, T^{n-1}\}$ via the "basic projections"?

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Answer (Dey–Easley–Monika 2025)

Yes! Every projection in $\text{comb}(T)$ is a sum of basic (Riesz) projections, uniquely indexed by a subset $S \subseteq \sigma(T) \setminus \{0\}$, forming a Boolean algebra with $2^{|\sigma_0(T)|}$ elements.

See `arXiv:2512.22497`.

Thank You!

Questions?

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